Section A-Research paper

THE UPPER EDGE-TO-EDGE GEODETIC NUMBER OF A GRAPH

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Abstract
An edge-to-edge geodetic set S in a connected graph G is called a *minimal edge-to- edge geodetic set* if no proper subset of S is an edge-to-edge geodetic set of G. The *upper edge-to-*edge *geodetic number* $g_{ee}^+(G)$ of G is the maximum cardinality of a minimal edge-to- *edge* geodetic set of G. The upper edge-to-edge geodetic number $g_{ee}^+(G)$ of a graph is studied and is determined for certain classes of graphs. It is shown that, for every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $g_{ee}(G)=a$ and $g_{ee}^+(G)=b$.
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1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [1]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an *$u - v$ geodesic*. The *eccentricity* $e(u)$ of a vertex u is defined by $e(u) = max\ \{d(u,v) : v \in V\}$.Each vertex in V at which the eccentricity function is minimized is called a *central vertex* of G and the set of all central vertices of G is called the *center* of G and is denoted by $Z(G)$.The *radius* r and *diameter* d of G are defined by $r = min\ \{e(v) : v \in V\}$ and $d = max\ \{e(v) : v \in V\}$ respectively. For subsets A and B of $V(G)$, the *distance*$d(A, B)$ is defined as $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. An $u - v$ path of length $d(A, B)$ is called an *$A - B$ geodesic* joining the sets A, B where $u \in A$ and $v \in B$. A vertex x is said to *lie* on an $A - B$ geodesic if x is a vertex of an $A-B$ geodesic. For $A = \{u, v\}$ and $B = \{z, w\}$ with uv and zw edges, we write an $A-B$ geodesic as $uv-zw$ geodesic and $d(A, B)$ as $d(uv, zw)$. A set $S \subseteq E$ is called an *edge-to-vertex geodetic set* if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *edge-to-vertex geodetic basis* of G. The edge-to-vertex geodetic number of a graph is introduced and studied in [6] and further studied in [8, 9]. The geodetic number of a graph is studied in [2,3, 4, 6]. A set $S \subseteq E$ is called an *edge-to-edge geodetic set* of G if every edge of G is an element of S or lies on a geodesic joining a pair of edges of S. The *edge-to-edge geodetic number* g_{ee}(G) of G is the minimum cardinality of its edge-to- edge geodetic sets and any edge-to- edge geodetic set of cardinality $g_{ee}(G)$ is said to be a g_{ee}-set of G.A double star is a tree with diameter three. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. The following theorems are used in sequel.

Theorem 1.1. [5] If v is an extreme vertex of a connected graph G, then every edge-to-edge geodetic set contains at least one extreme edge is incident with v.

Theorem 1.2. [5] For any non-trivial tree T with k end vertices, $g_{eve}(T) = k$.

Theorem 1.3. [5] For any connected graph G, $g_{ee}(G) = q$ if and only if G is a star.

2. The Edge-to-Edge Geodetic Number of a Graph

Definition 2.1. An edge-to-edge geodetic set S in a connected graph G is called a *minimal edge-to-edge geodetic set* if no proper subset of S is an edge-to-edge geodetic set of G. The *upper edge-to-* edge *geodetic number* $g_{ee}^+(G)$ of G is the maximum cardinality of a minimal edge-to- *edge* geodetic set of G.

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1v_6, v_3v_4\}$ is a minimum edge-to-edge geodetic set of G so that $g_{ee}(G) = 2$. The set $S_1 = \{v_1v_2, v_3v_4, v_5v_6\}$ is an edge-to-edge geodetic set of G and it is clear that no proper subset of $S_1$is an edge-to-edge geodetic set of G and so $S_1$is a minimal edge-to-edge geodetic set of G. Also it is easily verified that no four element or five element subset of edge set is a minimal edge-to-edge geodetic set of G, it follows that $g_{ee}^+(G) = 3$.

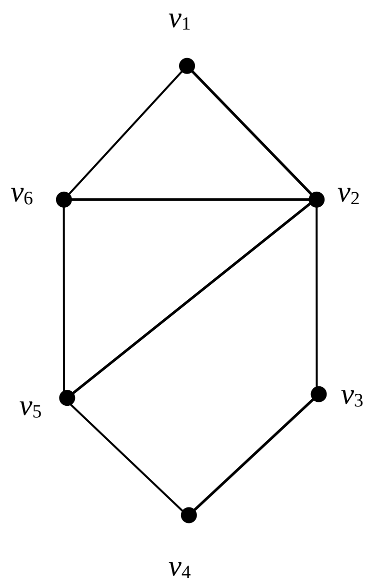


Figure 2.1

Remark2.3. Every minimum edge-to- *edge* geodetic set of G is a minimal edge-to-edge geodetic set of G and the converse is not true. For the graph G given in Figure 2.1, $S_1 = \{v_1v_2, v_3v_4, v_5v_6\}$ is a minimal edge-to-edge geodetic set but not a minimum edge-to-edge geodetic set of G.

Observation 2.4.

(i) Let G be a connected graph with cut-vertices and S an edge-to-edge geodetic set of G. Then every branch of G contains an element of S.

(ii) Let G be a connected graph with cut-edges and S an edge-to- edge geodetic set of G. Then for any non-pendant cut-edge e of G, each of the two components of $G - e$ contains an element of S.

(iii) Let G be a connected graph and S be a g_{ee}-set of G. Then no non-pendant cut-edge of G belongs to S.

Corollary 2.5. For any non-trivial tree T with k end-edges, $g_{ee}^+(T) = k$.

In the following we determine the upper edge-to- edge geodetic number of some standard graphs.

Theorem 2.6. For a complete graph $G = K_p (p \geq 4)$, $g_{ee}^+(G) = p - 1$.

Proof. Let S be any set of $p - 1$ adjacent edges of K_p incident at a vertex, say v. Since each edge of K_p is incident with an edge of S, it follows that S is an edge-to- *edge* geodetic set of G. If S is not a minimal edge-to-edge geodetic set of G, then there exists a proper subset S' of S such that S' is an edge-to-edge geodetic set of G. Therefore there exists at least one vertex, say u of K_p such that u is not incident with any edge of S'. Hence u is neither incident with any edge of S' nor lies on a geodesic joining a pair of edges of S' and so S' is not an edge-to-edge geodetic set of G, which is a contradiction. Hence S is a minimal edge-to-edge geodetic set of G. Therefore $g_{ee}^+(G) \geq p - 1$.Suppose that there exists a minimal edge-to-edge geodetic set M such that $|M| \geq p$. Since M contains at least p edges, $\langle M \rangle$ contains at least one cycle. Let $M' = M - \{e\}$, where e is an edge of a cycle which lies in $<M>$. It is clear that M' is an edge-to-edge geodetic set with $M' \subsetneq M$, which is a contradiction. Therefore, $g_{ee}^{+}(G) = p - 1$. ■

Theorem 2.7. For the complete bipartite graph $G = K_{m,n}$ $(2 \le m \le n)$, $g_{ee}^{+}(G) = n + m - 2$

Proof. Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be a bipartition of G. Let $S_i = \{x_iy_1, x_iy_2, ..., x_iy_{n-1}, x_1y_n, x_2y_n, \ldots, x_{i-1}y_n, x_{i+1}y_n, \ldots, x_my_n\}$, $(1 \le i \le m)$, $M_j = \{x_1y_j, x_2y_j, ..., x_{m-1}y_j, x_my_1, x_my_2, \ldots, x_my_{j-1}, x_my_{j+1}, \ldots, x_my_n\}$, $(1 \le j \le n)$ and $N_k = \{x_1y_1, x_2y_2, ..., x_{m-1}y_{m-1}, x_my_m, x_my_{m+1}, \ldots, x_my_n\}$ with $|S_i| = |M_j| = n + m - 2$ and $|N_k| = n$. It is easily verified that any minimal edge-to-edge geodetic set of G is of the form either S_i or M_j or N_k. Since no proper subset of $S_i (1 \le i \le m)$, M_j $(1 \le j \le n)$ and N_k is an edge-to-edge geodetic set of G, it follows that, $g_{ee}^{+}(G) = n + m - 2$. ■

THE EDGE-TO-EDGE GEODETIC NUMBER AND UPPER EDGE-TO- EDGE GEODETIC NUMBER OF A GRAPH

In this section, connected graphs G of size q with upper edge-to- *edge* geodetic number q or q-1 are characterized.

Theorem 2.8. For a connected graph G, $2 \le g_{ee}(G) \le g_{ee}^{+}(G) \le q$.

Proof. Any edge-to-edge geodetic set needs at least two edges and so $g_{ee}(G) \ge 2$. Since every minimal edge-to-edge geodetic set is an edge-to-edge geodetic set, $g_{ee}(G) \le g_{ee}^{+}(G)$. Also, since $E(G)$ is an edge-to-edge geodetic set of G, it is clear that $g_{ee}^{+}(G) \le q$. Thus $2 \le g_{ee}(G) \le g_{ee}^{+}(G) \le q$. ■

Remark 2.9. The bounds in Theorem 2.8 are sharp. For any non-trivial path P, $g_{ee}(P) = 2$. For any tree T, $g_{ee}(T) = g_{ee}^{+}(T)$ and $g_{ee}^{+}(K_{1,q}) = q$ for $q \ge 2$. Also, all the inequalities in the theorem are strict. For the complete graph $G = K_5$, $g_{ee}(G) = 3$, $g_{ee}^{+}(G) = 4$ and $q = 10$ so that $2 < g_{ee}(G) < g_{ee}^{+}(G) < q$.

Theorem 2.10. For a connected graph G, $g_{ee}(G) = q$ if and only if $g_{ee}^{+}(G) = q$.

Proof. Let $g_{ee}^{+}(G) = q$. Then $S = E(G)$ is the unique minimal edge-to-edge geodetic set of G. Since no proper subset of S is an edge-to- edge geodetic set, it is clear that S is the unique minimum edge-to- edge geodetic set of G and so $g_{ee}(G) = q$. The converse follows from Theorem 2.8. ■

Corollary 2.11. For a connected graph G of size q, the following are equivalent:

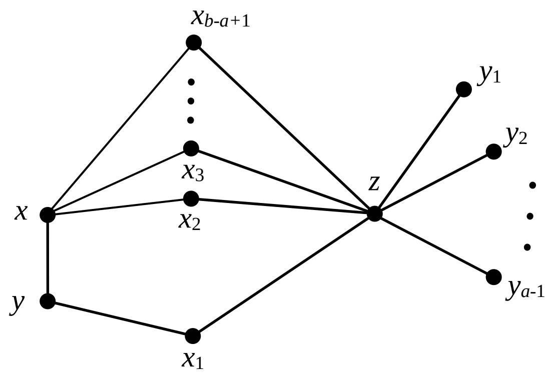
i) $g_{ee}(G) = q$
ii) $g_{ee}^{+}(G) = q$
iii) $G = K_{1,q}$.

Proof. This follows from Theorem 2.10 ■

Theorem 2.12. For every two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $g_{ee}(G) = a$ and $g_{ee}^{+}(G) = b$.

Proof. If $a = b$, let $G = K_{1,a}$. Then by Corollary 2.11, $g_{ee}(G) = g_{ee}^{+}(G) = a$. So, let $2\leq a< b$. Let P: x, y be a path on two vertices. Let G be the graph in Figure 2.2 obtained from P by adding new vertices z, x_1, x_2, …, x_{b-a+1},y_1, y_2,…, y_{a-1} and joining each vertex y_i $(1 \leq i\leq a - 1)$ and each vertex $x_i(1 \leq i\leq b - a + 1)$ with z, each vertex $x_i(2 \leq i\leq b - a + 1)$ with x andx_1 with y. Let $S = \{zy_1, zy_2, …, zy_{a-1}\}$ be the set of end-edges of G. Clearly, S is contained in every edge-to-edge geodetic set of G. It is clear that S is not an edge-to- edge geodetic set of G and so $g_{ee}(G) \geq a$. However $S' = S \cup \{xy\}$ is an edge-to- edge geodetic set of G so that $g_{ee}(G) = a$.

Now, $T = S \cup \{yx_1, xx_2, …, xx_{b-a+1}\}$ is an edge-to- edge geodetic set of G. We show that T is a minimal edge-to-edge geodetic set of G. Let W be any proper subset of T. Then there exists at least one edge say $e \in T$ such that $e \notin W$. First assume that $e = zy_i$ for some i $(1 \leq i \leq a - 1)$. Then the edge zy_i is neither incident with an edge of W nor lies on any geodesic joining a pair of edges of W and so W is not an edge-to- edge geodetic set of G. Now, assume that $e = xx_j$ for some j $(2 \leq j \leq b - a + 1)$. Then the edge xx_j is neither incident with an edge of W nor lies on a geodesic joining any pair of edges of W and so W is not an edge-to- edge geodetic set of G. Next, assume that $e = yx_1$. Then the edgeyx_1 is neither incident with an edge of W nor lies on a geodesic joining any pair of edges of W and so W is not an edge-to-edge geodetic set of G. Hence T is a minimal edge-to- edge geodetic set of G so that $g_{ee}^{+}(G) \geq b$. Now, we show that there is no minimal edge-to- edge geodetic set X of G with $|X| \geq b + 1$. Suppose that there exists a minimal edge-to- edge geodetic set X of G such that $|X| \geq b + 1$. Clearly, $S\subseteq X$. Since S' is an edge-to- edge geodetic set of G, it follows that $xy \notin X$. Let $M_1 = \{yx_1, xx_2, xx_3…, xx_{b-a+1}\}$ and $M_2 = \{zx_1, zx_2, zx_3…, zx_{b-a+1}\}$. Let $X = S \cup S_1 \cup S_2$, where $S_1 \subseteq M_1$ and $S_2 \subseteq M_2$. First we show that $S_1 \subsetneq M_1$ and $S_2 \subsetneq M_2$. Suppose that $S_1 = M_1$. Then $T\subseteq X$ and so X is not a minimal edge-to- edge geodetic set of G, which is a contradiction. Suppose that $S_2 = M_2$. If $yx_1 \notin X$, then y is neither incident with an edge of X nor lies on a geodesic joining any pair of edges of X and so X is not an edge-to- edge geodetic set of G, which is a contradiction. If $yx_1 \in X$ and if xy_i do not belong to S_1 for all i $(2 \leq i \leq b - a + 1)$, then x is neither incident with an edge of X nor lies on a geodesic joining any pair of edges of X and so X is not an edge-to- edge geodetic set of G, which is a contradiction. Therefore xx_i belong to S_1 for some i $(2 \leq i \leq b - a + 1)$. Without loss of generality let us assume that $xy_2 \in S_1$. Then $X' = X - \{zx_2\}$ is an edge-to- edge geodetic set of G with$X' \subsetneq X$, which is a contradiction. Therefore, $S_1 \subsetneq M_1$ and $S_2 \subsetneq M_2$. Next we show that $V(<S_1>) \cap V(<S_2>)$ contains no x_i $(1 \leq i \leq b - a + 1)$. Suppose that $V(<S_1) \cap V(<S_2)$ contains v_i for some i $(1 \leq i \leq b - a + 1)$. Without loss of generality let us assume that $y_2 \in V(<S_1) \cap V(<S_2)$. Then $X'' = X - \{zx_2\}$ is an edge-to-edge geodetic set of G with $X'' \subsetneq X$, which is a contradiction. Therefore $|S_1 \cup S_2| = b - a + 1$. Hence it follows that $|X| = a - 1 + b - a + 1 = b$, which is a contradiction to $|X| \geq b + 1$. Therefore $g_{ee}^{+}(G) = b$. ■



G

Figure 2.2

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