Analyze the carbon regulations that apply to a production system that integrates a process for reworking defective items within a finite planning horizon(FPH)

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3. Abstract:

Manufacturing companies are increasingly recognizing the importance of sustainable supply chain management (SSCM) and the need to address the issue of reworking defective items in their operations, both for their business goals and for the environment. To achieve sustainability goals, it is essential to design supply chain networks while taking ecological and environmental factors into account. This article's focus is on developing an environmentally friendly supply chain system within the framework of the emission trading mechanism. The problem at hand is formulated mathematically as a non-linear programming model with the objective of minimizing the total cost and carbon emissions. The approach is applied to analyse two distinct regulations: the cap and tax scheme under a finite planning horizon (FPH). Additionally, the study includes a sensitivity analysis, along with graphical or tabular representations, to enhance its effectiveness. The numerical results indicate the optimal solution for the decision variables, and a decrease in carbon emissions is observed. To solve the problem, a programming model in the Mathematica software version -12.0 is employed.

1. Introduction:

It is imperative for supply chain operations to maintain a reduction in carbon emissions while optimizing total costs. In reality, meeting customer demand is crucial for both systems and processes. With an increasing awareness among people, it has become essential to minimize carbon emissions. This research aims to demonstrate the impact of reducing carbon dioxide emissions while simultaneously optimizing costs and order quantity. As per Heather Lovell's

[2010] research, emission policies were established under the international climate change sovereignty. It is possible for businesses and individuals to offset their carbon dioxide emissions by participating in projects or initiatives aimed at reducing greenhouse gases anywhere in the world. These initiatives could involve the implementation of clean or renewable energy technologies or direct carbon capture from the atmosphere, such as through reforestation or tree planting efforts. In 1997, the Kyoto Protocol was established as a result of the United Nations Framework Convention on Climate Change (UNFCCC). The protocol introduced carbon offset regulations, which are referred to as project-based or Clean Development Mechanism (CDM) processes under the International Agreement. Dye and Yung (2014) have provided further explanations on this topic. The emission of greenhouse gases from supply chain processes is a critical concern, as stated by Zhou et al. [2021]. In response, many governments or regulatory agencies have imposed carbon taxes on industries, individuals, or organizations that exceed a predetermined emission cap. This measure aims to encourage energy conservation and decarbonization. Consequently, the financial impact of emission taxes on the supply chain and the management strategies that follow have become significant issues. In response, several national governments have attempted to reduce carbon dioxide emissions through various carbon reduction strategies, including carbon offsets, carbon taxes, cap-and-trade systems, carbon caps, and eco-sustainable technological requirements. The reduction of carbon emissions and optimization of total costs are critical for any supply chain operation. However, in the real world, both systems and processes are not fully realized without customer demand. As people's awareness of environmental issues continues to increase, the need to reduce carbon emissions becomes more pressing. This study aims to investigate the impact of reducing carbon dioxide emissions and optimizing costs and order quantities in supply chain operations. The development of emission policies was initiated under international climate change governance, as stated by Heather Lovell [2010]. This research focuses on analyzing an inventory supply chain problem under the carbon emission regulations mentioned earlier to advance the understanding of this research area.

In reality, the assumption that reworked processes are completely perfect may not be valid due to several factors such as deterioration of items, machinery wear and tear, and low-quality raw materials. These imperfections in the production process result in the production of defective products that cannot be ignored. To ensure that the production process meets the required quality standards, defective products undergo a reworking process. However, it is assumed that after reworking, the quality of the defective products is the same as that of newly manufactured products. RMISCP must operate effectively to increase revenue and remain competitive in an increasingly globalized world. SMMR sytem refers to an inventory supply chain problem that considers multiple retailers and a single manufacturer/supplier. While several studies have examined inventory management problems in SMMR system, none have yet explored the rework process with carbon emission regulations under FPH.

Additionally, the objective of the defective item rework process is to manage the inventory level during production and meet the demands of multiple retailers while minimizing the overall cost of the supply chain, while complying with carbon emission regulations. This research aims to examine the impact of carbon emission regulations on the inventory supply chain problem of a single manufacturer/supplier and multiple retailers (RMISCP) that includes the rework process under FPH.

The remaining sections of this research manuscript are structured as follows. Section 1 provides a literature review. subsequent section2. follow by a description of notations and assumptions. Section 3 focuses on the mathematical model, including the derivation of a non-linear differential equation. Section 4 presents a suitable algorithm, while Section 5 provides numerical examples and sensitivity analysis. Finally, Section 6 concludes the paper with managerial observations and a list of possibilities for future extensions.

2. Literature Review:

In recent years, there has been greater focus on studying inventory control or management with respect to carbon emissions. Governments have imposed limits on carbon emissions for individuals and companies, known as carbon caps. Those who exceed their carbon caps are subject to a carbon tax imposed by their government to reduce carbon emissions. Several research studies have been conducted in the context of inventory models with carbon emissions. Wahab et al. (2011) were the first to explore the effect of carbon emissions on supply chain inventory modeling, creating a mathematical framework for identifying optimal replenishment/shipping cycles and manufacturing costs. Benjaafar et al. (2013) introduced a model that amended traditional models to include strategic planning that considered both cost and environmental impact by incorporating carbon pollution variables into decision parameters.Dye and Yung (2014) formulated a model that discussed carbon regulation with credit-dependent demand. Hovelaque and Bironneau (2015) evaluated an economic quantity order model in which the product's deterministic demand rate was determined by each year's carbon emissions and its price. Micheli et al. (2018) developed a model that considered emission policies such as carbon caps, taxes, and carbon offsets, and discussed their environmental and economic impacts. According to their research, carbon emissions produced by transportation contribute significantly to the carbon footprint. Tao and Xu (2019) investigated the impact of carbon regulatory policies and customers' low-carbon awareness on optimized order quantity, overall costs, and CO2 emissions using an EOQ-type model. Rodrigues Dias et al. (2022) explained a model in which they considered carbon offsets with carbon emission suitability, and discussed global awareness about sustainability. Ramudhin et al. (2013) incorporated carbon emissions and the total cost of the supply chain into supply chain planning, solving them through a multi-objective mixed-integer linear programming model. Cheng et al. (2016) studied the effect of carbon emission laws on the traditional inventory transportation problem and implemented linearization methods and a hybrid genetic algorithm to build and run mixed integer nonlinear programming models within both infinite and finite planning horizons.

Carbon emissions have become a topic of greater concentration and have a significant impact on the supply chain process. Therefore, carbon emissions are considered in this research work within a finite planning horizon (FPH).

The literature on product rework during production is vast, particularly regarding supply chain inventory control models. In this context, we will mention only a few studies that suggest new and different approaches to the rework process. Benkherouf et al. (2016), Bazan et al. (2016), and Vikramjeet Singh et al. (2019) investigated inventory management systems that included manufacturing and remanufacturing. However, little research has been done on optimizing supply chain processes with rework under a finite planning horizon (FPH). Lakdere et al. (2016) investigated an inventory management system that included manufacturing, remanufacturing, and renovating activities over a finite planning horizon, assuming demand would vary over time, and that the buyer would return used goods, which would then be categorized as "manufacturable" or "renovative" after inspection. Vikramjeet Singh et al. (2019) provided a comprehensive solution for re-manufacturing a product in a supply chain model with time-dependent quadratic demand, shortages, Weibull deterioration, and partial inventory backlog. Additionally, they discussed trade credit between the supplier and the retailer. While there is an extensive literature on inventory control and management models with a product rework process, very few research studies explore the implications of carbon emissions on the rework or remanufacturing process in production FPH.

This research evaluates a supply chain system that coordinates manufacturing, repair, and remanufacturing activities to meet time-varying demand under a carbon tax regulatory system. Konstantara et al. [2021] have created a mixed integer nonlinear programming problem to determine the optimal strategy. Lakdere et al. [2016] investigated an inventory management system with manufacturing, remanufacturing, and renovating activities over a finite planning horizon, assuming that demand varies over time and returned goods are inspected and categorized as "manufacturable" or "renovative". Vikramjeet Singh et al. [2019] provide a comprehensive solution for remanufacturing a product in a model of supply chains with time-dependent quadratic demand, shortages, Weibull deterioration, and partial inventory backlog, and also consider trade credit between the supplier and the retailer. While there has been extensive literature on inventory control and management models with rework or remanufacturing processes, only a few research studies explore the effect of carbon emissions on such processes. This paper focuses on remanufacturing as a way to reduce environmental problems in the context of carbon tax policies.The RMISCP approach was also used to investigate the impact of emission reduction measures on inventory decisions. Zouadi et al. (2018) presented a manufacturing/remanufacturing approach that reduces carbon emissions. The model incorporates emissions from production, remanufacturing, and transportation and examines their effects. Samuel et al. (2020) suggested a single supplier and multiple retailers inventory model and investigated the effect of carbon emissions on supply chain activities. They compared the performance of a carbon cap and a carbon cap and trade on the single supplier and multiple retailers inventory approach. Modak and Kelle (2021) developed a single-manufacturer and multiple-retailer inventory model with a carbon tax scheme. The objective is to maximize total profit by simultaneously evaluating the investment in recycling, item cost, purchasing quantity, and contribution quantity. Furthermore, Gu et al. (2021) proposed a single-manufacturer and multiple-retailer inventory model that includes carbon emissions and transportation systems. They focused on reducing carbon emissions and economic costs while improving customer satisfaction throughout the supply network. The environmental and economic attributes of the supply chain are the most researched aspects throughout the literature review. However, only a few studies address remanufacturing, carbon cap and offset, and all of the aforementioned components of supply chain sustainability with multiple retailers and a single manufacturer under a finite planning horizon (FPH).

2.1. Problem defining:

The present research study introduces a finite planning horizon lot sizing **SMMR** inventory control model that incorporates manufacturing, rework, or remanufacturing activities under carbon constraints. The main target of this carbon mechanism is to evaluate the production, rework, and repairing scheme that reduces both producer and retailers' total costs, including production order costs, set-up costs, production costs, penalty costs, deteriorating costs, and emission costs. This is achieved through a nonlinear programming problem (NLPP) solution.

2.2. Research gap

The environmental and economic attributes of the supply chain are probably the most researched aspects throughout the review of the literature. Only a handful of research studies accommodate all the components of supply chain sustainability, including remanufacturing, carbon cap, and offset, within a finite planning horizon (FPH). However, very few researchers have examined the rework process with carbon emission policies for an SMMR inventory supply chain system over the finite planning horizon with unequal cycle lengths.

3. Assumptions and Notations:

The analysis was conducted based on the following assumptionsand Notations.

1. It is assumed that a collection or set contains only one item or element in cases where a single-item collection is presumed.
2. The study has a well-defined planning horizon that is finite but not fixed
3. The study assumes that the lead time is zero for the entire duration.
4. According to the framework, neither retailers nor producers are allowed to have shortages.
5. Production of stock is based on demand in this context.
6. This research involves a single manufacturer and multiple retailers.
7. The rate of deterioration of the inventory remains constant.
8. Shipments from the producer to the retailer happen at irregular intervals.

Listed below are the parameters relevant for the Producer, who also serves as the supplier:	
$\mathbf{I_{P_i}(t)}$	The inventory level of the manufactured goods for a specific time period, denoted as "t," during the cycle that follows the "i+1" cycle.
$\mathbf{P}$	The number of units of a product that can be produced within a given unit of time.
Ss	Production setup cost, including even other transportation ($/lot).
$\boldsymbol{H_s}$	The cost incurred by the supplier for holding inventory, measured in dollars per

	unit and per unit of time.
C_p	The cost of penalty that the supplier incurs.
Ps	The cost incurred by the supplier for purchasing raw materials and services from the another supplier.
R	Rate of % at which products are remanufactured.
TCs	The total cost incurred by the supplier over the entire planning horizon H.
Listed below are the parameters relevant for the retailers:	
O_r	The cost of ordering goods from supplier, measured in dollars per unit and per unit of time.
H_r	The cost of storing goods at the buyer's agent's facility, measured in dollars per unit and per unit of time.
θ	The parameter represents the rate of deterioration of items at the retailers,and supplier site, where $0 < \theta < 1$.
$D_p and D_r$	The total cost of deterioration incurred by both the retailers and supplier, measured in dollars per unit.
a	The initial annual rate of demand for a product or service.
b	The annual rate at which the demand for a product or service is increasing.
I_{b_i}	The inventory level of a retailer's stock for a specific time period, denoted as "t," during the cycle that follows the "i+1" cycle.
TC_r	The total cost incurred by the retailer over the entire planning horizon H.
Listed below are the decision variables:	
n	The number of shipments made to the business customer (retailer) during each phase.
H	A predetermined annually time period.
t_i	The ith replenish time, assuming $t_i = 0$ and t_n = H
Ti+1	The duration of the (i+1)th replenishment period, where i = 0, 1, 2, ..., expressed as $T_{i+1} = t_{i+1} - t_i$
Listed below are variables related to transportation and carbon emissions:	
C	The limit/cap on the amount of carbon emissions that is allowed.
τ	The cost associated with emitting carbon, measured in dollars.
$\widehat{P}_r$	The quantity of carbon emissions generated as a result of placing an order. (unit/time/ton)
$\widehat{P}_s$	The quantity of carbon emissions generated as a result of placing an order from supplier site. (unit/time/ton)
$\widehat{H_s}$	The amount of carbon emissions generated due to the supplier's holding of inventory. (unit/time/ton)
$\widehat{H_r}$	The amount of carbon emissions generated due to the retailer's holding of inventory. (unit/time/ton)
C_e	The overall quantity of carbon emissions produced by the refrigeration system.(ton CO_2/unit)
v_c	The variable cost of fuel used during transportation by the retailers is dependent on the amount of fuel consumed. ($/litre)
e_2	The additional cost incurred by the retailer for refrigerated transportation of one unit of item, including carbon emissions.

	($/unit/km)
$\mathbf{e_1}$	The amount of carbon dioxide emissions charged for the retailer's transportation. ($/km)
$\mathbf{C_1}$	Fuel consumption of the retailer's vehicle when empty. (litres per kilometre)
$\mathbf{C_2}$	Additional transportation energy consumption by retailers in term of refrigeration and vehicle services for every per ton of payload (litre/km/ton)
d	Distance covered by transportations from supplier to each retailer. (km)
$\mathbf{F_c}$	Fixed transportation cost for retailers when order is placed by them. ($)

4. Mathematical nonlinear problem formulation focused on a carbon cap:

This section presents a mathematical inventory model for a single-manufacturer and multiple retailers (SMMR) system that incorporates remanufacturing and carbon regulations. A carbon cap policy is considered in each period of time, which means that carbon dioxide emissions must not exceed the specified limit (cap) within a finite planning horizon (FPH). This limit may be imposed by an external regulatory agency or government to comply with specific emission limits. The model consists of three production phases, as shown in Figure 1.

4.1. Phase 1. $t_i < t < \boldsymbol{t_i'}$

$$\frac{dI_{P_i}(t)}{dt} = \phi \sum_{j=1}^{5} D_j\,(\mathrm{t}) - \theta * I_{P_i}(t)(\mathbf{1})$$

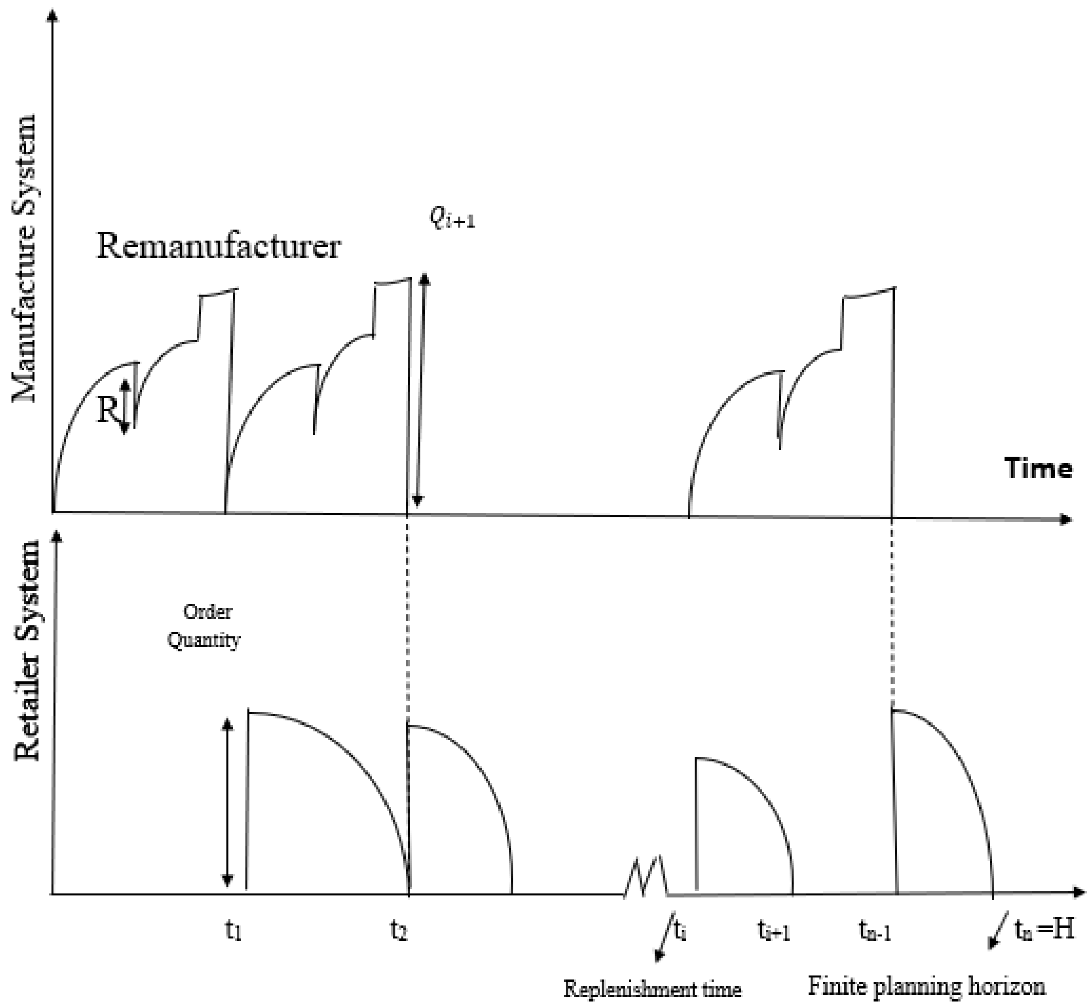
$\sum_{j=1}^{5} D_j\,(\mathrm{t}) = \mathrm{D(t)} = \mathrm{a} + \mathrm{b*t}$

$$\frac{dI_{P_i}(t)}{dt} = \phi D(t) - \theta * I_{P_i}(t)$$

$$\frac{dI_{P_i}(t)}{dt} + \theta * I_{P_i}(t) = \phi\,(a + b * t) \quad (\mathbf{2})$$

$$t_i < t < t_i^{'}$$

$$I_{P_i}(t) = e^{-\theta t}\phi \int_{t_i}^{t} (a + b * u) * e^{\theta * u} \quad (\mathbf{3})$$



$$I_{P_i}(t) = \phi \int_{t_i}^{t} (a + b * u) * e^{\theta(u-t)}\, du (\mathbf{4})$$

Figure 1 Production level of supplier and Replenishment to retailers.

$$I_{P_i}(t_i^{'}) = I_{P_{(1,i)}} \text{and} \quad I_{P_i}(t_i) = 0 (\mathbf{5})$$

$$I_{P_{(1,i)}} = Inventory\ level\ at\ point\ 'A'\ is\ I_{P_i}(\boldsymbol{t_i^{'}}) = \phi \int_{t_i}^{t_i^{'}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{'})}\,\mathrm{du}$$

$$I_{P_{(1,i)}} = \phi \left\{ \frac{-b+a\theta+b\theta \boldsymbol{t_i^{'}} + e^{\theta(-\boldsymbol{t_i^{'}}+t_i)}(b-a\theta-b\theta t_i)}{\theta^2} \right\} (\mathbf{6})$$

4.2. Phase 2. For $t_i^{'} < t < t_i^{''}$

After Remanufacturing, inventory level at point B is $= I_{P_i}(t_i^{'}) - R\%\ of\ I_{P_i}(t_i^{'})$

$I_{P_{(2,i)}} =$ Inventory level at point 'c' is $I_{P_i}(t_i^{''}) = I_{P_i}(t_i^{'}) - R\%\ of I_{P_i}(t_i^{'}) + \phi \int_{t_i^{'}}^{t_i^{''}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-ti'')}$du*IP(2,i)=IPiti′−R% ofIPiti′+*ϕ*ti′ti″*(a + b∗u)*∗eθ(u−ti″)*du

$$I_{P_{(2,i)}} = I_{P_i}(t_i^{'}) - R\%\ of I_{P_i}(t_i^{'}) + \phi \int_{t_i^{'}}^{t_i^{''}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{''})}\,\mathrm{du}$$

$$I_{P_{(2,i)}} = \phi \int_{t_i}^{t_i^{'}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{'})}\,\mathrm{du} - R\%\ of \phi \int_{t_i}^{t_i^{'}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{'})}\,\mathrm{du} + \phi \int_{t_i^{'}}^{t_i^{''}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{''})}\,\mathrm{du} (\mathbf{7})$$

4.3. Phase 3. For $t_i^{''} < t < t_{i+1}$

Inventory level at the point D= $I_{P_i}(t_i^{''}) + R\%\ of\ I_{P_i}(\boldsymbol{t_i^{'}})$

$$I_{P_i}(t_i^{'}) - R\%\ of\ I_{P_i}(t_i^{'}) + \phi \int_{t_i^{'}}^{t_i^{''}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{''})}\,\mathrm{du} + R\%\ of\ I_{P_i}(t_i^{'})$$

$$\phi \int_{t_i}^{t_i^{'}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{'})}\,\mathrm{du} - R\%\ of\ \phi \int_{t_i}^{t_i^{'}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{'})}\,\mathrm{du} + \phi \int_{t_i^{'}}^{t_i^{''}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{''})}\,\mathrm{du} + R\%\ of \phi \int_{t_i}^{t_i^{'}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i^{'})}\,\mathrm{du}$$

$$\phi \int_{t_i}^{t_i'} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i')}\,\mathrm{du} + \phi \int_{t_i'}^{t_i''} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i'')} \quad (\mathbf{8})$$

Inventory level at the point 'E' is

$$I_{P_{(3,i)}} = \text{Inventory level at 'D'} + \phi \int_{t_i''}^{t_{i+1}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t)}\,\mathrm{du}$$

$$I_{P_{(3,i)}} = \phi \int_{t_i}^{t_i'} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i')}\,\mathrm{du} + \phi \int_{t_i'}^{t_i''} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_i'')}\,\mathrm{du} + \phi \int_{t_i''}^{t_{i+1}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_{i+1})}\,\mathrm{du}$$

$$I_{P_i}(t_i < t < t_{i+1}) =$$

$$\frac{ab(-3 + e^{\theta(t_i'-t_i'')} + e^{\theta(-t_i'+t_i)} + e^{\theta(t_i''-t_{1+i})} - \theta t_i'(-1 + e^{\theta(t_i'-t_i'')}) - \theta t_i''(-1 + e^{\theta(t_i''-t_{1+i})}) - \theta t_i e^{\theta(t_i-t_i')}}{\theta^2}$$

Total cost of production= setup +holding +penalty + purchasing +carbon emission cost + deterioration cost

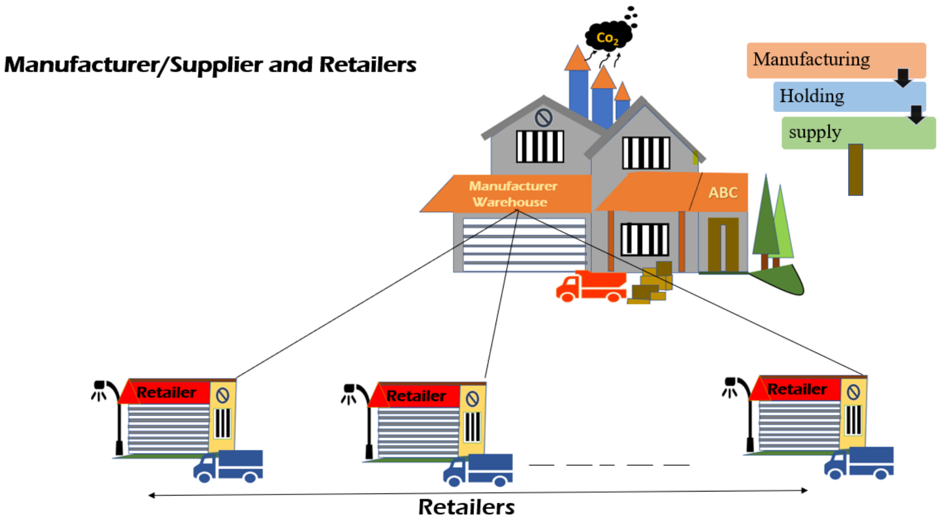
$$\left[TC_s\right. = n*S_s + H_s\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} I_{p_i}(t)dt + c_p * R\% \, of I_{P_i}(\boldsymbol{t_i'}) + \text{Ps} * \sum_{i=1}^{n} I_{P_i}(t_i < t < t_{i+1})$$
$$+ \sum_{i=0}^{m_1-1} D_p * \theta\int_{t_i}^{t_{i+1}} I_{p_i}(t)dt$$
$$+ \sum_{i=0}^{n-1}\tau\left(C - \left(\widehat{P_s} * I_{P_i}(\boldsymbol{t_i'}) + \widehat{H_s}\int_{t_i}^{t_{i+1}} \phi\int_{t_i}^{t} (\text{a} + \text{b}*\text{u}) * e^{\theta(u-t)}\,\text{du}t\right)\right)\Bigg]\Bigg[TC_s$$
$$= n*S_s$$
$$+ \sum_{i=1}^{n}\Bigg((H_s + \theta D_p)\phi\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} (\text{a} + \text{b}*\text{u}) * e^{\theta(u-t)}\,\text{du}\,dt + c_p$$
$$* R\% \, of \phi\int_{t_i}^{t_i'} (\text{a} + \text{b}*\text{u}) * e^{\theta(u-t_i')}\,\text{du} + \text{Ps}$$
$$* \frac{ab(-3 + e^{\theta(t_i'-t_i'')} + e^{\theta(-t_i'+t_i)} + e^{\theta(t_i''-t_{1+i})} - \theta t_i'(-1 + e^{\theta(t_i'-t_i'')}) - \theta t_i''(-1 + e^{\theta(t_i''-t_{1+i})}) - \theta t_i e^{\theta(t_i-t_i'}}{\theta^2}$$

$$\sum_{i=0}^{n-1}\tau\left(C - \left(\widehat{P_s} * \int_{t_i}^{t_i'} (\text{a} + \text{b}*\text{u}) * e^{\theta(u-t_i')}\,\text{du} + \widehat{H_s}\int_{t_i}^{t_{i+1}} \phi\int_{t_i}^{t} (\text{a} + \text{b}*\text{u}) * e^{\theta(u-t)}\,\text{du}dt\right)\right)$$

$$\Bigg[TC_s = n * S_s + \sum_{i=1}^{n} \Bigg((H_s + \theta D_p - \tau * \widehat{H_s})\phi \int_{t_i}^{t_{1+i}} \frac{-b + a\theta + b\theta t + e^{\theta(-t+t_i)}(b - a\theta - b\theta t_i)}{\theta^2} dt + (c_p * R\% - \widehat{P_s} * \tau)\phi \int_{t_i}^{t_i'} (a + b*u) * e^{\theta(u-t_i')} du + Ps * \Bigg(\phi \int_{t_i}^{t_i'} (a + b*u) * e^{\theta(u-t_i')} du + \phi \int_{t_i'}^{t_i''} (a + b*u) * e^{\theta(u-t_i'')} du + \phi \int_{t_i''}^{t_{i+1}} (a + b*u) * e^{\theta(u-t_{i+1})} du \Bigg)\Bigg)\Bigg]$$

$$TC_s = n * S_s + \sum_{i=1}^{n} \Bigg((H_s + \theta D_p)\phi \int_{t_i}^{t_{1+i}} \frac{-b + a\theta + b\theta t + e^{\theta(-t+t_i)}(b - a\theta - b\theta t_i)}{\theta^2} dt + (c_p of R\% + Ps - \widehat{P_s} * \tau)\phi \int_{t_i}^{t_i'} (a + b*u) * e^{\theta(u-t_i')} du + Ps * \Bigg(\phi \int_{t_i'}^{t_i''} (a + b*u) * e^{\theta(u-t_i'')} du + \phi \int_{t_i''}^{t_{i+1}} (a + b*u) * e^{\theta(u-t_{i+1})} du \Bigg)\Bigg) + \tau * C$$

$$t_i' = 0.1 + t_i and t_i'' = 0.1 + t_i' = 0.2 + t_i$$



$$TC_s = n * S_s + \sum_{i=1}^{n} \Bigg((H_s + \theta D_p - \tau * \widehat{H_s})\phi \int_{t_i}^{t_{1+i}} \frac{-b + a\theta + b\theta t + e^{\theta(-t+t_i)}(b - a\theta - b\theta t_i)}{\theta^2} dt + (c_p of R\% + \mathrm{Ps} - \widehat{P_s} * \tau)\phi \int_{t_i}^{0.1+t_i} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-0.1-t_i)} \mathrm{du} + \mathrm{Ps} * \Bigg(\phi \int_{0.1+t_i}^{0.2+t_i} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-0.2-t_i)} \mathrm{du} + \phi \int_{0.2+t_i}^{t_{i+1}} (\mathrm{a} + \mathrm{b} * \mathrm{u}) * e^{\theta(u-t_{i+1})} \mathrm{du} \Bigg) \Bigg) + \tau * \boldsymbol{C} \quad (10)$$

$$\frac{\partial(TC_s)}{\partial t_i} = (H_s + \theta D_p - \tau * \widehat{H_s})\phi\left\{\frac{-b + a\theta + b\theta t_1 + e^{\theta(-t_1+t_{i-1})}(b - a\theta - b\theta t_{i-1})}{\theta^2} + \int_{t_i}^{t_{i+1}} \frac{e^{\theta(-t+t_i)}(-bt_i) + (b - a\theta - b\theta t_i)e^{\theta(-t+t_i)}}{\theta} dt\right\} + (c_p of R\% + Ps - \widehat{P_s} * \tau)\phi * \left\{(-\theta)\int_{t_i}^{0.1+t_i} (a + b * u) * e^{\theta(u-0.1-t_i)} du + [a + b * (0.1 + t_i)] - (a + b * t_i) * e^{-\theta(0.1)}\right\} + Ps * \phi\left\{(-\theta)\int_{0.1+t_i}^{0.2+t_i} (a + b * u) * e^{\theta(u-0.2-t_i)} du + [a + b * (0.2 + t_i)] - [a + b * (0.1 + t_i)] * e^{-\theta(0.1)} - [a + b * (0.2 + t_i)]e^{\theta(0.2+t_i-t_{i+1})}\right\} \quad (\mathbf{11})$$

It is a approach of running a business in which Supplier/manufacturer supply their items to end users through a number of different retailers.

Retailer's solution:

$$\frac{dI_{b_i}(t)}{dt} = -\sum_{j=1}^{5} D_j(t) - \theta * I_{b_i}(t)$$

$$\frac{dI_{b_i}(t)}{dt} = -D(t) - \theta * I_{b_i}(t) \quad (\mathbf{12})$$

$$\sum_{j=1}^{5} D_j(t) = \sum_{j=1}^{5} a_j + b_j t = a + bt$$

$$t_i < t < t_{i+1}$$

$$I_{b_i}(t) = e^{-\theta t} \int_{t}^{t_{i+1}} (a + bu)e^{\theta * u}\, \mathrm{dt}$$

$$I_{b_i}(t_{i+1}) = 0\text{and} \quad I_{b_i}(t_i) = \int_{t_i}^{t_{i+1}} (a + bu)e^{\theta *(u-t)}\, \mathrm{dt}$$

$$I_{b_i}(t) = \int_{t}^{t_{i+1}} (a + bu)e^{\theta(u-t)}\, \mathrm{du}$$

$$I_{b_i}(t) = \left[\frac{(a + bu)e^{\theta(u-t)}}{\theta} - \frac{b}{\theta^2} e^{\theta(u-t)}\right]_{t}^{t_{i+1}}$$

$$I_{b_i}(t) = \frac{(a + bt_{i+1})e^{\theta(t_{i+1}-t)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t)} - \frac{(a + bt)}{\theta} + \frac{b}{\theta^2} \quad (\mathbf{13})$$

$$The\ order\ quantity\ for\ \ i^{th} cycles$$

$$Q_{i+1} = I_{b_i}(t_i) = \int_{t_i}^{t_{i+1}} (a + bt)e^{\theta(t-t_i)}\, \mathrm{dt}$$

$$Q_{i+1} = I_{b_i}(t_i) = \left[\frac{(a + bu)e^{\theta(u-t_i)}}{\theta} - \frac{b}{\theta^2} e^{\theta(u-t_i)}\right]_{t_i}^{t_{i+1}}$$

$$Q_{i+1} = I_{b_i}(t_i) = \frac{(a + bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t_i)} - \frac{(a + bt_i)}{\theta} + \frac{b}{\theta^2} (\mathbf{14})$$

$$O_r = \sum_{j=1}^{5} O_{rj}, H_r = \sum_{j=1}^{5} H_{rj}, D_r = \sum_{j=1}^{5} D_{rj}, C_e = \sum_{j=1}^{5} C_{ej}, F_c = \sum_{j=1}^{5} F_{cj}, d = \sum_{j=1}^{5} d_j, v_c = \sum_{j=1}^{5} v_{c_j}, C_1 = \sum_{j=1}^{5} C_{1_j}, C_2 = \sum_{j=1}^{5} C_{2_j}, e_1 = \sum_{j=1}^{5} e_{1_j}, e_2 = \sum_{j=1}^{5} e_{2_j}$$

Ordering costs of the buyer$= n * O_r$

Holding cost = $H_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} I_{b_i}(t)\, dt$

$H_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} e^{-\theta t} \int_{t}^{t_{i+1}} (a_i + b_i t)e^{\theta * u} du\ \ dt$**(15)**

Deteriorating costs according to Sarkar et al.2012 $D_r = \sum_{j=1}^{5} D_{rj} = D_{r1} + D_{r2} + D_{r3} + D_{r4} + D_{r5}$

$$D_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \theta * I_{b_i}(t)\, dt$$

$$D_r * \theta \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \left\{ \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)}\, \mathrm{du} \right\} \mathrm{dt}$$

$D_{c_b} * \theta \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)}\, \mathrm{du} \quad dt$ **(16)**

Amount of carbon emission due to transportation:

$T_c =$
$F_c + 2\mathrm{d}v_c C_1 + \mathrm{d} * v_c C_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)}\, dt + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)}\, dt$ **(17)**

Amount of Carbon emission according to She et al .209 and Mishra N.K, Ranu [2022]. Which include, the first T_c amount carbon emission due to transportation and other factors) associated with replenishment order, second $\hat{P}_r$ associated with inventory replenished and third $\widehat{h_r}$ associated with handling of inventory (refrigeration effect).

$$C_e = \sum_{i=0}^{n-1} \hat{P}_r * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)} du\, dt + F_c + 2\mathrm{d}v_c C_1 + \mathrm{d} * v_c C_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)}\, dt + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)}\, dt$$

Cost of carbon emission:

$\sum_{i=0}^{n-1} \tau \Bigg(C - \Big(\hat{P}_r * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)} du\, dt + F_c + 2\mathrm{d}v_c C_1 + \mathrm{d} *$
vcC2titi+1(*a*+ b t) *e*θt−t*idt*+2d *e1*+d *e2titi+1*(*a*+ b t) *e*θt−t*idt* **(18)**

Retailer's overall cost

Total cost =Ordering cost + Inventory holding cost + Deteriorating cost +Carbon emission cost + transportation cost

$$TC_r = n * O_r + H_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} I_{b_i}(t)dt + D_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \theta * I_{b_i}(t)dt + \sum_{i=0}^{n-1} \tau \left(C - \left(\hat{P_r} * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)} du\, dt + F_c + 2\mathrm{d}v_c C_1 + \mathrm{d} * v_c C_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)} dt + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)} dt \right) \right) \quad (\mathbf{19})$$

$$TC_r = n * O_r + H_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)} \mathrm{du}\, dt + \theta D_r \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)} \mathrm{du}\, dt + \sum_{i=0}^{n-1} \tau \left(C - \left(\hat{P_r} * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu)e^{\theta(u-t)} du\, dt + F_c + 2\mathrm{d}v_c C_1 + \mathrm{d} * v_c C_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)} dt + 2\mathrm{d}\, e_1 + \mathrm{d}\, e_2 \int_{t_i}^{t_{i+1}} (a + \mathrm{b\,t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)} dt \right) \right)$$

$$TC_r = n * O_r + \{H_r + \theta * D_r + \widehat{h_r}\} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(u-t)} \, \mathrm{du} \, dt + \{\mathrm{d}\tau e_2 - \hat{P}_r * \tau - \mathrm{d} * v_c C_2 \tau\} \int_{t_i}^{t_{i+1}} (a + \mathrm{b}\,\mathrm{t}) e^{\theta(\mathrm{t}-\mathrm{t}_i)} dt + \tau C - \tau F_c - 2\tau \mathrm{d} v_c C_1 - 2\tau \mathrm{d}\, e_1 \quad \textbf{(20)}$$

$$TC_r = n * O_r + \left\{\frac{H_r + \theta * D_r + \widehat{h_r}}{\theta} + \mathrm{d}\tau e_2 - \hat{P}_r * \tau - \mathrm{d} * v_c C_2 \tau\right\} \int_{t_i}^{t_{1+i}} ((a + \mathrm{bt}) e^{\theta(t-t_i)} + \tau C - \tau F_c - 2\tau \mathrm{d} v_c C_1 - 2\tau \mathrm{d}\, e_1 - \left(\frac{H_r + \theta * D_r + \widehat{h_r}}{\theta}\right) (a * H + 0.5\, b * H^2)$$

Lemma1:

t_i increase where i=1,2,3………… n-1 strictly monotonic increase function of last replenishment cycle t_n

In this lemma we can see a bonding among replenishment time, last replenishment time, length of replenishment time and time horizon.

$T_{i+1} = t_{i+1} - t_i$ and $t_n = \mathrm{H}\text{-}T_n$

Theorem 1: The unique solution for the nonlinear system of Eqn (11) is the optimal replenishment time period for a fixed replenishment cycle n. The Hessian matrix of TC_smust be positive definite for ti to be the minimum for a fixed n. Therefore, as established in Appendix A, the theorem states that TC_s is positive definite. As a result, the optimal value of ti for a given fixed positive integer n can be computed using numerical iterative techniques and Mathematica version 12.0 programs. Based on the optimal value of t_i, the total cost function will also be optimal.

Proof: See below.

Theorem 2: TC_s (n, t_0, t_1,……….., tn) is a convex function for no of replenishment cycles in a finite horizon planning H.

5. Methodology for solving problems:

In this subsection, we introduce an approach for determining the option solution with the minimum retailer cost. Looking at the preceding section's challenge, the minimal total cost of retailer occurs at point $t_i$simultaneously, satisfied the $\frac{\partial(TC_s)}{\partial t_i}=0$. We derive the following equations by the first partial derivative of TC_s w.t.r to t_i, respectively.

$$\frac{\partial(TC_s)}{\partial t_i} = (H_s + \theta D_p - \tau * \widehat{H_s})\phi\left\{\frac{-b + a\theta + b\theta t_1 + e^{\theta(-t_1+t_{i-1})}(b - a\theta - b\theta t_{i-1})}{\theta^2} + \int_{t_i}^{t_{i+1}} \frac{e^{\theta(-t+t_i)}(-bt_i) + (b - a\theta - b\theta t_i)e^{\theta(-t+t_i)}}{\theta} dt\right\} + (c_p ofR\% + \text{Ps} - \widehat{P_s} * \tau)\phi * \left\{(-\theta)\int_{t_i}^{0.1+t_i} (\text{a} + \text{b} * \text{u}) * e^{\theta(u-0.1-t_i)}\,\text{du} + [\text{a} + \text{b} * (0.1 + t_i)] - (\text{a} + \text{b} * t_i) * e^{-\theta(0.1)}\right\} + \text{Ps} * \phi\left\{(-\theta)\int_{0.1+t_i}^{0.2+t_i} (\text{a} + \text{b} * \text{u}) * e^{\theta(u-0.2-t_i)}\,\text{du} + [\text{a} + \text{b} * (0.2 + t_i)] - [\text{a} + \text{b} * (0.1 + t_i)] * e^{-\theta(0.1)} - [\text{a} + \text{b} * (0.2 + t_i)]e^{\theta(0.2+t_i-t_{i+1})}\right\} == 0$$

To solve the defined problems, we construct an iterative approach that is based on the method developed by Wu and zho (2014)

$$\frac{\partial^2(TC_s)}{\partial {t_i}^2} = (c_p ofR\% + \text{Ps} - \widehat{P_s} * \tau)\phi\left(b - be^{-0.1\theta} - \frac{b(1.\theta - 1.e^{-0.1\theta}\theta)}{\theta}\right) + \text{Ps} * \phi\left(b - be^{-0.1\theta} - be^{\theta(0.2+ti-ti+1)} - b(1.\theta - 1.e^{-0.1\theta}\theta)\theta - e^{\theta(0.2+ti-ti+1)}\theta(a+b(0.2+ti)\right) + (Hs + \theta Dp - \tau * \widehat{H_s})\phi\left(\frac{b\theta - e^{\theta(t_{i-1}-t_i)}\theta(b - a\theta - b\theta t_{i-1})}{\theta} - \frac{b(1 - e^{\theta(t_i - t_{1+i})} + \theta - e^{\theta(t_i - t_{i+1})}\theta t_i + 2a\theta^3 t_i - a\theta^3 t_{i+1})}{\theta}\right) \quad \textbf{(A)}$$

$$\frac{\partial^2 TC_s}{\partial t_i \partial t_{i-1}} = \frac{(H_s + \theta D_p - \tau * \widehat{H_s}) * \phi * (-b e^{\theta(t_{i-1}-t_i)}\theta + e^{\theta(t_{i-1}-t_i)}\theta(b - a\theta - b\theta t_{i-1}))}{\theta^2} \quad \textbf{(B)}$$

Similarly

$$\frac{\partial^2 TC_s}{\partial t_i \partial t_{i+1}} = \frac{b(H_s + \theta D_p - \tau \ * \widehat{H_s}) * \phi(-\theta + (e^{\theta(t_i - t_{i+1})}\theta - a\theta^3)t_i)}{\theta^2} + e^{\theta(0.2 + t_i - t_{i+1})}\text{Ps} * \phi * \theta(a + b(0.2 \ + t_i))$$

(C)

$$\frac{\partial^2 TC_s}{\partial t_i \partial t_n} = 0 \qquad \textbf{(D)}$$

for all n ≠ i, i+1, i-1

Furthermore,

$$\nabla^2 TC_s = \begin{bmatrix} \frac{\partial^2 TC_s}{\partial t_1^2} & \frac{\partial^2 TC_s}{\partial t_1 \partial t_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial^2 TC_s}{\partial t_2 \partial t_1} & \frac{\partial^2 TC_s}{\partial t_2^2} & \frac{\partial^2 TC_s}{\partial t_2 \partial t_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2 TC_s}{\partial t_3 \partial t_2} & \frac{\partial^2 TC_s}{\partial t_3^2} & \frac{\partial^2 TC_s}{\partial t_3 \partial t_4} & 0 & 0 & 0 & 0 & 0 \\ \dots. & \dots. & \dots & \dots & \dots & \dots. & \dots. & \dots. & \dots. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 TC_s}{\partial t_{n_1-1} \partial t_{n_1-2}} & \frac{\partial^2 TC_s}{\partial t_{n_1-1}^2} & \frac{\partial^2 TC_s}{\partial t_{n_1-1} \partial t_{n_1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 TC_s}{\partial t_{n_1} \partial t_{n_1-1}} & \frac{\partial^2 TC_s}{\partial t_{n_1}^2} \end{bmatrix}$$

Figure 2 Hessian matrix .

TC_s is positive definite if Eq. A, B, C and D satisfy the given inequality.

$\frac{\partial^2 TC_s}{\partial t_i^2} \geq |\frac{\partial^2 TC_s}{\partial t_i t_{i-1}}| + |\frac{\partial^2 TC_s}{\partial t_i t_{i+1}}|$ or

$$\frac{\partial^2 TC_s}{\partial t_i^2} - |\frac{\partial^2 TC_s}{\partial t_i t_{i-1}}| - |\frac{\partial^2 TC_s}{\partial t_i t_{i+1}}| \geq 0$$

$$(H_s + \theta D_p - \tau * \widehat{H_s})\phi\left(\frac{b\theta - e^{\theta(t_{i-1}-t_i)}(\theta-1)(b - a\theta - b\theta t_{i-1}) - b(-e^{\theta(t_i - t_{i+1})}\theta t_i + 2a\theta^3 t_i - a\theta^3 t_{i+1}) - (-be^{\theta(t_{i-1}}}{\theta} - \right) + (c_p of R\% + \mathrm{Ps} - \widehat{P_s} * \tau)\phi\left(b - be^{-0.1\theta} - \frac{b(1.\theta - 1.e^{-0.1\theta}\theta)}{\theta}\right) + \mathrm{Ps} * \phi\left(b - be^{\theta(0.2 + t_i - t_{i+1})} - \frac{b(1.\theta)}{\theta}\right) > 0$$

that is true for all i= 1, 2, . . ., n

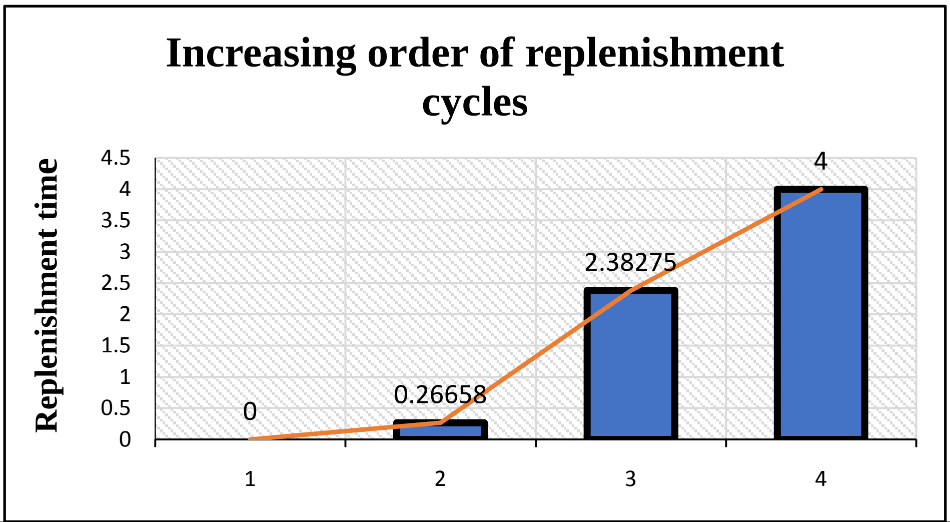
Moreover, the Hessian matrix had to be positive definite since it contains positive diagonal members and has strictly diagonal dominating features. As a result, the optimal replenishment interval to the nonlinear system of Equation (11) is obtained. now we need to show that the optimal solution of the non-linear equation (11) is unique and also $\mathrm{TC_s}(\mathrm{t_i}, \mathrm{n})$ is optimal function throughout the optimal value of $\mathrm{t_i}$ in a finite horizon planning H.

Furthermore, because it had strictly diagonal dominating characteristics and positive diagonal members, the Hessian matrix required to be positive definite. As a result, the optimum replenishment interval for nonlinear system Equation (11) is established. Now we need to demonstrate the convexity of $\mathrm{TC_s}(\mathrm{t_i}, \mathrm{n})$ throughout the optimal value of ti in the finite horizon planning H.

6. Numerical example to solve this problem:

Fundamental values of all parameters with their appropriate units are listed here. $O_b =$ 80 \$/order,H_r=1 \$/unit/time,$\Theta = 3, H_s =, Ps = 3$, a= 50, 60, 70 , b=15, $D_p =, \widehat{h_r} =, D_r =, Ss = 100$,cˆ,$\hat{P_r} = 0.02$,$\hat{P_s} = 0.02, R =, \tau = 6$,$C_p = 0.01$, $\tau = 1.3$, $\mathrm{v_c} = 8$, $\mathrm{e_2} = 2.31 \times 10^{-6}$, $\mathrm{e_1} = 0.043$,$\mathrm{C_1} = 25$,$\mathrm{C_2} = 0.36$, d=25, $\mathrm{F_c} = 0.01$, C= , d= , ϕ =To find the values of $\mathrm{t_i}$, cost function, solved the non-linear solution are solved by iterative numerical approach in "Mathematica" mathematical problem solving software. **Table 10.1, Table 10.2, Figure 10.2** and **Figure 10.3** Give a comprehensive information about the optimal supplier/manufacturer's overall cost for a= 0.5, 0.125, and 0.625 are $ **299171.9,** $**299146.0**, **and** $ **299182.3**are reach its optimal level at 2, 2, and 3 optimal replenishment cycles respectively. After achieving its minimum at n=2, 2, and 3 then again goes upward gradually for all upcoming cycles. **Table 10.1**, and **Table 10.2** reveals the convexity behaviour of supplier/manufacturer's overall cost function.This is also informed with the help of graphical explanation.

Table 1. Optimal Planning time for replenishment.



↓ a	→ t_i	t_0	t_1	t_2	t_3	n	Q_{nt}	Tc_s	Tc_r
0.5		**0**	0.34754	4.		**2**	12.867(	235.542	**299171.9**
0.125		0	0.34769	4.		2	11.0627	227.561	**299146.0**
		0	0.26658	2.38275	4.	3	□12.16	234.438	**299182.3**

Figure 2demonstrates that the increasing duration for replenishment orders to be placed.

Table 2 Revel the optimal value of the Tc_r, Tc_s *and* Q_{nt} .

↓ a	→ m_1	1	2	3	4	5	6	7
0.5		482240.9	**299171.9**	299374.0	299519.7	299615.8	299669.6	299701.7
0.125		386245.8	**299146.0**	299356.0	299508.6	299610.4	299668.7	299698.5
0.625		439306.7	299295.**5**	**299182.3**	299523.3	299617.5	299669.9	299702.9

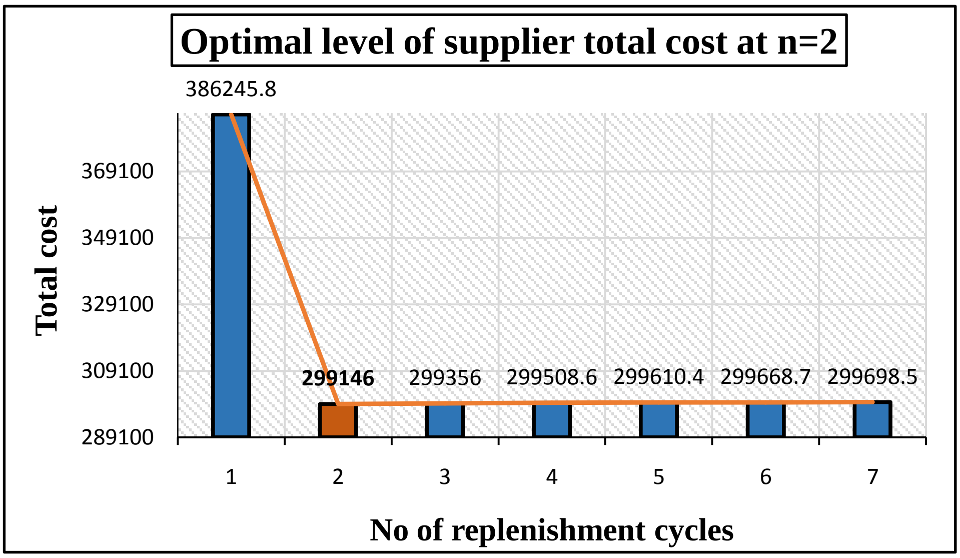
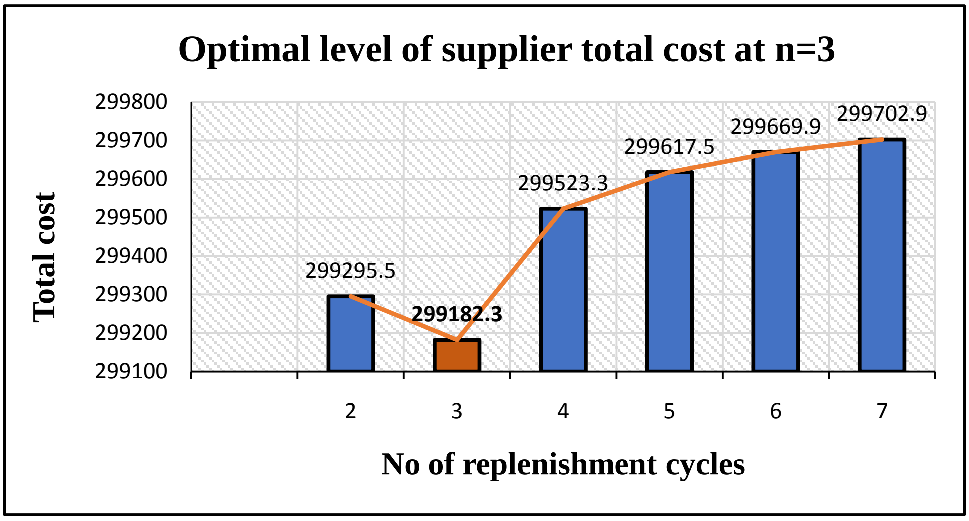


Figure 3. displays the ideal level of supplier cost during the second replenishment cycle.

Figure 4. Demonstrates the supplier cost level that is considered ideal for the second replenishment cycle.



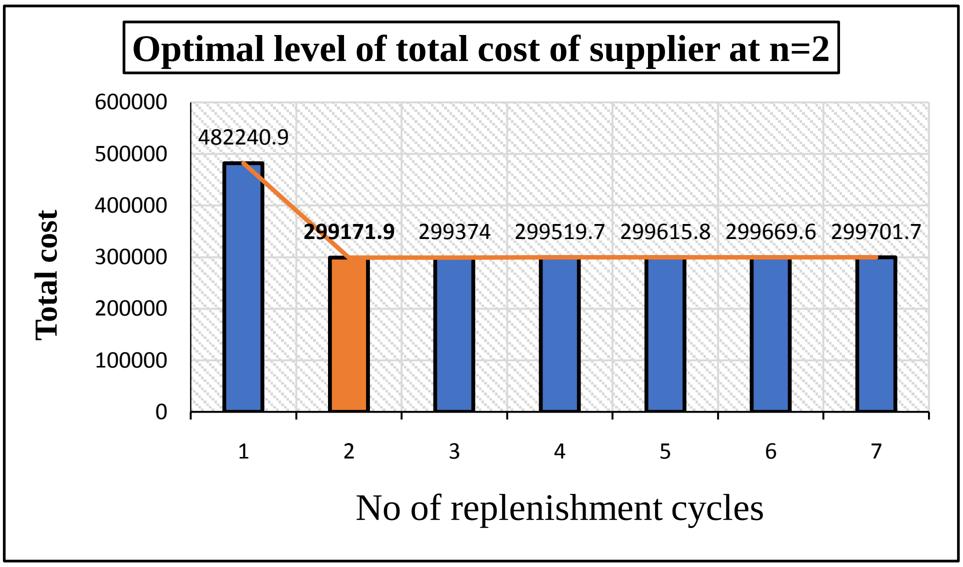


Figure 5.shows the supplier cost level that is considered optimal for the third replenishment cycle.

Table3 Identify the following results of detailed sensitivity analysis.

Parameters	%Changes	Optimal Replenish cycle	Total order Quantity Q_{nt}	Total cost of Retailer Tc_r	Total cost of supplier Tc_s
A	−50	2	11.0627	238.438	299154.8
	−25	2	11.0627	227.563	299146.0
	0	2	12.8670	235.542	299171.9
	+25	2	12.1533	234.438	299180.3
	+50	2	11.4236	233.289	299188.6
B	−50	2	16.5956	241.454	704181.5
	−25	2	11.4701	239.055	731207658.4
	0	2	12.8670	235.542	142299.34
	+25	2	13.9324	231.081	2057841.7
	+50	2	14.6733	225.692	3279.5
O_b	−50	2	12.8729	115.559	139382.69
	−25	2	12.8729	155.559	139382.69
	0	2	12.8729	235.559	139382.69
	+25	2	12.8729	295.559	139382.69
	+50	2	12.8729	355.559	139382.69
Ss	−50	2	12.8729	235.559	142274.3
	−25	2	12.8729	235.559	142261.8
	0	2	12.8729	235.559	142299.3
	+25	2	12.8729	235.559	142311.8
	+50	2	12.8729	235.559	142324.3
Ps	−50	2	6.5223	225.711	3325482.3
	−25	2	9.4323	225.711	3773490.4
	0	2	12.8729	235.559	142299.3
	+25	2	14.5015	240.221	267132.5
	+50	2	15.5373	243.186	19467.1
R	−50	2	12.8560	235.511	2298744.9
	−25	2	12.8588	235.519	92515.8
	0	2	12.8729	235.559	142299.3
	+25	2	12.8869	235.599	93256.9
	+50	2	12.9007	235.639	216987.4
τ	−50	2	4.8525	241.543	81420066113.8
	−25	2	9.3831	225.570	120296869840.5
	0	2	12.8729	235.559	142299.3
	+25	2	12.4557	234.365	18203511.0
	+50	2	12.4557	234.365	18203511.0
ϕ	−50	2	25.9945	235.559	1987220.9
	−25	2	25.9945	235.559	993620.9
	0	2	25.9945	235.559	142299.3
	+25	2	25.9945	235.559	4968020.7
	+50	2	25.9945	235.559	5961620.7
$D_p and D_r$	−50	2	19.9083	229.403	172261836.5
	−25	2	17.4554	226.769	199438281.3
	0	2	25.9945	235.559	142299.3
	+25	2	29.6750	239.086	90732502.2
	+50	2	33.7843	242.906	63556057.4

H_s	−50 −25 0 +25 +50	2 2 2 2 2	11.2515 12.0282 25.9945 13.7306 14.6546	230.918 233.142 235.559 238.014 240.659	12227015.9 2902677.4 142299.3 1884162.5 3477456.4
H_r	−50 −25 0 +25 +50	2 2 2 2 2	12.8670 12.8670 12.8670 12.8670 12.8670	225.532 229.287 235.559 236.542 238.423	142299.4 142299.4 142299.4 142299.4 142299.4
$\widehat{h_r}$	−50 −25 0 +25 +50	2 2 2 2 2	12.8670 12.8670 12.8670 12.8670 12.8670	225.032 230.287 235.559 240.798 265.790	142299.4 142299.4 142299.4 142299.4 142299.4

6.1.Sensitivity analysis

In this paragraph, we provide a piece of information about the sensitivity analysis to explore the performances of the proposed framework when a set of different parameters is altered. At a particular time, only one parameter alters while all remine constant. Most important parametric value of the $\boldsymbol{O_r}$,$\boldsymbol{H_r}$,$\boldsymbol{\Theta}$, $\boldsymbol{H_s}$, $\boldsymbol{Ps}$, a, b, $\boldsymbol{D_p}$, $\boldsymbol{D_r}$, $\boldsymbol{Ss}$,**c**ˆ,$\boldsymbol{\widehat{P}_r}$,$\boldsymbol{\widehat{P}_s}$, $\boldsymbol{R}$, $\boldsymbol{\tau}$, $\boldsymbol{\tau}$, and **ϕ**are altered one by one. We are concentrating on analysing the fluctuation in all parameters on order quantity, total retailer cost, and supplier cost. The evaluation is based on the numerical example and prepared algorithm described earlier. Table 3 performs an evaluation and pictorial representation of the Replenishment order quantity, retailer and supplier cost parameters are changed from -50% to +50%, and also fluctuations in replenishment quantity, and expected total cost of retailer and supplier are noticed.

The main motive of this research analysis is to identify significant management application and

its important and applicability of the model's solutions. This evolution verified with the mathematical problem-solving software “MATHEMATICA” version-12 and iterative numerical mathematical approach for non-linear differential equation. Table 3reveals the comprehensive analysis in detail.

1. As **Table 3** is clearly identifying, supplier’s total cost $TC_s$is very sensitive to the parametric value of b, R, τ, H_s, D_p, P_s, ϕand D_s. moderately sensitive to S_s , a. and practically insensitive to H_r, $\widehat{h_r}$ and O_r

2. **Table** 3 give an idea of detail analysis that retailer's total cost TC_r is very sensitive to the parameters O_r, moderately sensitive to a, b, R, τ, H_s, D_p, P_s, ɸand $D_s$and practically insensitive to S_s.

3. The comprehensive and very clear study in **Table 3** shows that he orders quantity replenished by supplier to the retailer's Q_{nt} is very sensitive to the parameters b, τ, H_s, D_p, P_s, ɸand D_s, moderately sensitive to a , R and practically insensitive to O_r, ɸ, $\widehat{h_r}$, S_s and H_r .
4. No of replenishment cycles 'n' is shown in **Table 3** to be constant reactive for the parameters like a, b, τ, H_s, D_p, P_s, ɸ, H_r, D_s, τ,$\widehat{h_r}$, $S_s$and O_r,

7. Conclusion:

In this article, we begin by developing an inventory model under the assumption that the demand for a single item follows a linear function of time with unequal cycle lengths over a finite planning horizon (FPH). We also highlight collaborative supply chain management, where the framework is implemented to reduce total supply chain costs while controlling CO2 emissions. The main objective of the model is to reduce costs for both retailers and suppliers while also managing emission levels to protect the environment.

We use a numerical example and sensitivity analysis to demonstrate the effectiveness and properties of the proposed framework. Finally, we provide managerial recommendations to demonstrate the practical application of our proposed model.

In the future, we plan to extend this model to consider multi-item, inflation, shortage, and quadratic demand, among other factors, if necessary.

8. Managerial implications

Based on my understanding of the situation, my managerial suggestion would be to conduct a thorough analysis of the carbon regulations that are applicable to a production system that involves a rework process for defective production within a finite planning horizon (FPH). This would involve examining the potential environmental impact of the production process, identifying areas where improvements could be made to reduce carbon emissions, and developing strategies to ensure compliance with relevant regulations. It may also be useful to explore the use of more sustainable materials and production methods, as well as implementing measures to reduce waste and increase energy efficiency. The ultimate objective is to establish an economically viable production system that is also environmentally responsible.

The availability of this research study is essential for optimizing real-time supply chain management, which can aid in inventory control and enhance operational capacity. The study also recommends industry collaboration to support firms in utilizing this information to reduce costs.

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