Analysis of a Markovian Retrial Queue with Recurrent Customers, Switch overtime and Server Vacation

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Abstract

In this article, we examine analysis of a Markovian retrial queue with recurrent customers, switch over time and server vacation. In this model, retrial times, vacation times and service times are assumed to have an exponential distribution. We obtain the probability generating function for the number of customers in the system. We also compute the average number of customers in the system. Some of the special cases are discussed. To calibrate the model'sstability some numerical examples are illustrated.

Keywords: Retrial queue, Recurrent customers, Switch over time, Server vacation and Steady-state equations.

MSC 2010 No.:6K25, 68M20, 90B22

1. INTRODUCTION

In this paper, analysis of a Markovian retrial queue with recurrent customers, switch over time and server vacation is taken. Whenever the system becomes empty, the server leaves from the regular service period and goes on a vacation, but in a switch over time model, the server waits for some arbitrary amount of time before going to vacation. The switch over time model studied by Doshi[7].

Retrial queues are characterized by the phenomenon that arriving customers who find the server busy join the retrial group (called orbit) to repeat their request for service after some random time. Retrial queuing systems have beenwidely used to model many practical problems in telephone switching systems, telecommunication networks, and computers competing to gain service from a central processing unit. For recent bibliographies on retrial queues, see [2,3,4].

Boxma[5] and Cohen[6] studied an $M/G/1$ queue in which there is a fixed number of permanent customers present who rejoin the queue on their completion of service. This system with permanent customers in the retrial context was analyzed by Farahmand[11]. Moreno[17] discussed an $M/G/1$ retrial queue with recurrent customers and general retrial times. Queues with vacations have been studied extensively in the past: a comprehensive survey can be found in Kalyanaraman. R. and Pazhani Bala Murugan.S[14], Teghem[18] and Doshi[7].

The organization of this paper is as follows: The model under consideration is described in section 2. In section 3, we analyze the model by deriving the system steady state equations. Using the equations, the probability generating function of queue length are obtained in section 4.

2. Model Description

We consider an $M/M/1$ retrial queue with transit (also called ordinary) customers and a fixed number K ($K \geq 1$) of recurrent (also called permanent) customers. After service completion, recurrent customers always return to the retrial group and transit customers leave the system forever.

Transit customers arrive according to a Poisson process with rate λ. If a transit customers finds the server free on his arrival, he occupies the server, otherwise, he enters the re-trail group in accordance with an FCFS discipline. We will assume that only the transit customers at the head of the orbit are allowed for access to the server. Successive inter re-trial times of any transit customer follow an exponential distribution with rate α. The service times for the transit customers are exponentially distributed with rate μ_1.

There is a fixed number K of permanent customers in the system. After having received service, recurrent customers immediately return to the retrial group in accordance with an FCFS discipline. We will assume that only the recurrent customer at the head of orbit is allowed for access to the server. Successive inter retrial times of any recurrent customer follow an exponential distribution with rate β. The service times for the recurrent customers are exponentially distributed with rate μ_2. After each service completion, the nextcustomers to be served is determined by a competition between the retrial time of transit customers and recurrent customers.

After completion of service there is no transit customers in the orbit the server waits for an arbitrary period of time, which follows an exponential distribution with a rate γ. After completion of the waiting time the server takes a vacation of random length. At the end of a vacation, if the server finds no transit customer in the orbit, he immediately takes another vacation and continuous in this manner until he finds atleast one transit customer upon returnfrom vacation. The vacation times are the exponentially distributed with parameter θ. The inter arrival times, retrial times, service times and vacation times are mutually independent.

Let $O(t)$ be the number customers in the orbit at time t and $C(t)$ denotesthe server state at time t. The different possible states of the server are given below:

$$C(t) = \begin{cases} 0 & \text{if the server is idle (or)free (or) uncopied} \\ 1 & \text{if the server is busy with transit customers} \\ 2 & \text{if the server is busy with recurrent customers} \\ 3 & \text{if the server is on vacation} \end{cases}$$

We observe that $\{(O(t), C(t)) : t \geq 0\}$ is a continuous Markov chain.

3. Model Analysis

We define the following limiting probabilities for our subsequent analysis of the queueing model.

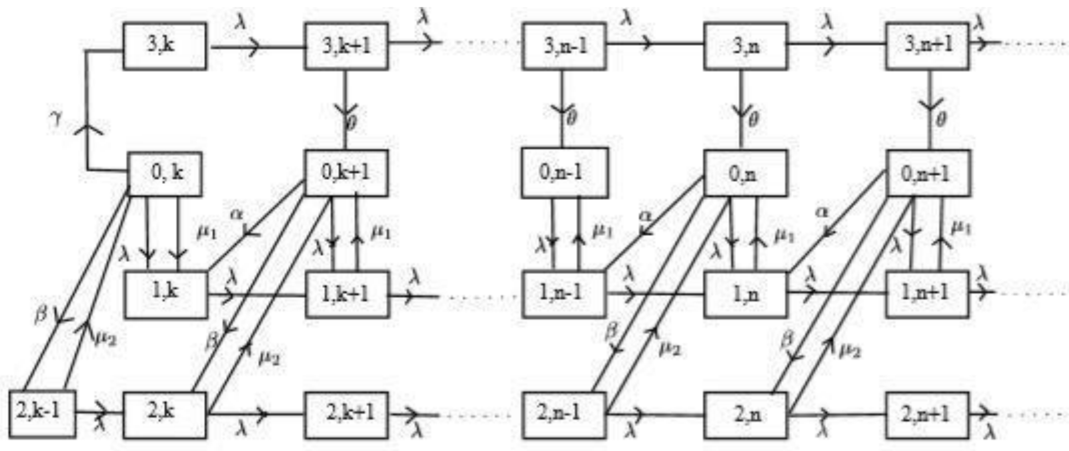

Figure 1:Transition state diagram of the system

$P_{0,n} \lim_{t\to\infty}\{C(t) = 0, O(t) = n\}, \quad n \geq k + 1$

$P_{1,n} \lim_{t\to\infty}\{C(t) = 1, O(t) = n\}, \quad n \geq k$

$P_{2,n} \lim_{t\to\infty}\{C(t) = 2, O(t) = n\}, \quad n \geq k$

$P_{3,n} \lim_{t\to\infty}\{C(t) = 3, O(t) = n\}, \quad n \geq k + 1$

The system has the following set of steady state equations:

$$\lambda P_{3,k} = \gamma P_{0,k}; \quad n = k \tag{1}$$

$$(\lambda + \theta)P_{3,n} = \lambda P_{3,n-1}; \quad n \geq k + 1 \tag{2}$$

$$(\lambda + \beta + \gamma)P_{0,k} = \mu_1 P_{1,k} + \mu_2 P_{2,k-1}; \quad n = k \tag{3}$$

$$(\lambda + \alpha + \beta)P_{0,n} = \mu_1 P_{1,n} + \mu_2 P_{2,n-1} + \theta P_{3,n}; \quad n \geq k + 1 \tag{4}$$

$$(\lambda + \mu_1)P_{1,k} = \lambda P_{0,k} + \alpha P_{0,k+1}; \quad n = k \tag{5}$$

$$(\lambda + \mu_1)P_{1,n} = \lambda P_{0,n} + \alpha P_{0,n+1} + \lambda P_{1,n-1}; \qquad n \geq k+1 \tag{6}$$

$$(\lambda + \mu_2)P_{2,k-1} = \beta P_{0,k}; \qquad n = k \tag{7}$$

$$(\lambda + \mu_2)P_{2,n} = \beta P_{0,n+1} + \lambda P_{2,n-1}; \qquad n \geq k \tag{8}$$

4. Probability Generating Functions of Queue Length

We define the following probability generating functions:

$$\left.\begin{aligned} P_0(z) &= \sum_{n=k}^{\infty} P_{0,n} z^n \\ P_1(z) &= \sum_{n=k}^{\infty} P_{1,n} z^n \\ P_2(z) &= \sum_{n=k-1}^{\infty} P_{2,n} z^n \\ P_3(z) &= \sum_{n=k}^{\infty} P_{3,n} z^n \end{aligned}\right\} \tag{9}$$

applying (9) into equations (2)-(8), we get

$$(\lambda + \alpha + \beta)P_0(z) = \mu_1 P_1(z) + \mu_2 z P_2(z) + \theta P_3(z) - (\lambda + \theta)P_{3,k} z^k + \alpha P_{0,k} z^k \tag{10}$$

$$(\lambda - \lambda z + \mu_1) z P_1(z) = (\lambda z + \alpha) P_0(z) - \alpha P_{0,k} z^k \tag{11}$$

$$(\lambda - \lambda z + \mu_2) z P_2(z) = (\beta) P_0(z) \tag{12}$$

$$(\lambda - \lambda z + \theta) P_3(z) = (\lambda + \theta) P_{3,k} z^k \tag{13}$$

substituting equations (11), (12) and (13) in (10), we get

$$P_0(z) = \frac{\begin{matrix}\lambda(\lambda+\theta)P_{3,k}z^{k+1}(\lambda-\lambda z+\mu_1)(\lambda-\lambda z+\mu_2) - \alpha\,\overline{\gamma}^1 P_{3,k} z^k \\ \times(\lambda-\lambda z+\theta)(\lambda-\lambda z+\mu_2)(\lambda z-\mu_1)\end{matrix}}{-(\lambda-\lambda z+\theta)\begin{bmatrix}(\lambda+\alpha+\beta)\lambda z(\lambda-\lambda z+\mu_1+\mu_2) \\ -(\lambda z+\alpha)\lambda\mu_1 - \beta\lambda z\mu_2 - \alpha\mu_1\mu_2\end{bmatrix}} \tag{14}$$

Substituting (14) in (11), we get

$$P_1(z) = \frac{-(\lambda+\theta)(\lambda z+\alpha)(\lambda-\lambda z+\mu_2)\lambda P_{3,k} z^{k+1} - \alpha\,\overline{\gamma}^1 P_{3,k} z^k (\lambda-\lambda z+\theta)(\lambda-\lambda z+\beta+\mu_2)}{-z(\lambda-\lambda z+\theta)\begin{bmatrix}(\lambda+\alpha+\beta)\lambda z(\lambda-\lambda z+\mu_1+\mu_2) \\ -(\lambda z+\alpha)\lambda\mu_1 - \beta\lambda z\mu_2 - \alpha\mu_1\mu_2\end{bmatrix}} \tag{15}$$

Substituting (14) in (12), we get

$$P_2(z) = \frac{-(\lambda+\theta)(\lambda-\lambda z+\mu_1)\lambda\beta P_{3,k} z^{k+1} - \alpha\,\overline{\gamma}^1 P_{3,k} z^k (\lambda-\lambda z+\theta)(\mu_1-\lambda z)}{-z(\lambda-\lambda z+\theta)\begin{bmatrix}(\lambda+\alpha+\beta)\lambda z(\lambda-\lambda z+\mu_1+\mu_2) \\ -(\lambda z+\alpha)\lambda\mu_1 - \beta\lambda z\mu_2 - \alpha\mu_1\mu_2\end{bmatrix}} \tag{16}$$

From (13), we get

$$P_3(z) = \frac{(\lambda+\theta)P_{3,k}z^k}{(\lambda-\lambda z+\theta)} \tag{17}$$

Let us define $P(z) = P_0(z) + z(P_1(z) + P_2(z)) + P_3(z)$ the pgf for number of customers in the system.

$$P(z) = \frac{\begin{matrix}-(\lambda+\theta)P_{3,k}z^k(\lambda z+\alpha)\mu_1(\lambda-\lambda z+\mu_2)+\alpha\bar{\gamma}^1\lambda P_{3,k}z^k \\ \times(\lambda-\lambda z+\theta)(\lambda-\lambda z+\beta+\mu_2)(\lambda z-\mu_1+z)\end{matrix}}{(\lambda-\lambda z+\theta)\begin{bmatrix}(\lambda+\alpha+\beta)\lambda z(\lambda-\lambda z+\mu_1+\mu_2) \\ -(\lambda z+\alpha)\lambda\mu_1-\beta\lambda z\mu_2-\alpha\mu_1\mu_2\end{bmatrix}} \tag{18}$$

Applying the normalizing condition $P(1) = 1$, in equation (18), we get

$$P_{3,k} = \left[\frac{\theta[\beta\lambda\mu_1+(\lambda+\alpha)\lambda\mu_2-\alpha\mu_1\mu_2]}{(\lambda+\theta)(\lambda+\alpha)\mu_1\mu_2+\alpha\bar{\gamma}^1\lambda\theta(\mu_2+\beta)(\lambda-\mu_1+1)}\right] \tag{19}$$

Which implies that the utilization factor is $\rho = \frac{\frac{\lambda}{\mu_1}\left(1+\frac{\beta\mu_1}{(\lambda+\alpha)\mu_2}\right)}{\frac{1}{1+\frac{\lambda}{\alpha}}}$ and the steady state condition is therefore

$$\frac{\frac{\lambda}{\mu_1}\left(1+\frac{\beta\mu_1}{(\lambda+\alpha)\mu_2}\right)}{\frac{1}{1+\frac{\lambda}{\alpha}}} \le 1 \tag{20}$$

Particular cases:

i) If $\theta \to \infty$ then the present model will be remodeled as an $M/M/1$ retrial queue with recurrent customers.

ii) If $\alpha \to \infty, \theta \to \infty$ then the present model will be remodeled as an $M/M/1$ retrial queue with recurrent customers.

iii) If $\alpha \to \infty, \theta \to \infty$ and k=0 then the present model will be remodeled as an $M/M/1$ queue.

iv) If $k = 0$ then the present model will be remodeled as an $M/M/1$retrial queue with server vacation.

v) If $k = 0$ and $\theta \to \infty$ then the present model will be remodeled as an $M/M/1$ retrial queue.

5. Operating Characteristics

Let E(L) denote the mean number of customers in the system. The Probability generating function for the number of customers in the system is

$$P(z) = \frac{N(z)}{D(z)} P_{3,k}$$

Differentiating with respect to z

$$P(z) = \frac{D(z)N'(z) - D'(z)N(z)}{[D(z)]^2} P_{3,k}$$

$$P_{3,k} = \left[\frac{1}{1+\frac{\lambda}{\theta}}\right]\left[\frac{1}{1+\frac{\lambda}{\alpha}} - \frac{\lambda}{\mu_1}\left(1 + \frac{\beta\mu_1}{(\lambda+\alpha)\,\mu_2}\right)\right]$$

$$\begin{aligned} N(z) = {} & (\lambda+\theta)(\lambda z+\alpha)\mu_1(\lambda-\lambda z+\mu_2)z^k \\ & +\alpha\,\overline{\gamma}^1\lambda(\lambda-\lambda z+\theta)(\lambda-\lambda z+\beta+\mu_2)(\lambda z-\mu_1+z)z^k \end{aligned}$$

$$\begin{aligned} N'(z) = {} & -(\lambda+\theta)\mu_1\ [\lambda(\lambda-\lambda z+\mu_2)z^k + (\lambda z+\alpha)\lambda z^k \\ & + (\lambda z+\alpha)(\lambda-\lambda z+\mu_2)kz^{k-1}\,] \\ & -\alpha\,\overline{\gamma}^1\lambda[(\lambda-\lambda z+\beta)(\lambda z-\mu_1+z)\lambda z^k \\ & + (\lambda-\lambda z+\theta)(\lambda z-\mu_1+z)\lambda z^k \\ & - (\lambda-\lambda z+\theta)(\lambda-\lambda z+\beta+\mu_2)(\lambda+1)z^k \\ & - (\lambda-\lambda z+\theta)(\lambda-\lambda z+\beta+\mu_2)(\lambda z-\mu_1+z)kz^{k-1}] \end{aligned}$$

$$D(z) = (\lambda-\lambda z+\theta)\begin{bmatrix}(\lambda+\alpha+\beta)\lambda z(\lambda-\lambda z+\mu_1+\mu_2) \\ -(\lambda z+\alpha)\lambda\mu_1 - \beta z\lambda\mu_2 - \alpha\mu_1\mu_2\end{bmatrix}$$

$$D'(z) = \begin{bmatrix}(-\lambda)[(\lambda+\alpha+\beta)\lambda z(\lambda-\lambda z+\mu_1+\mu_2) \\ -(\lambda z+\alpha)\lambda\mu_1 - \beta z\lambda\mu_2 - \alpha\mu_1\mu_2] \\ +(\lambda-\lambda z+\theta)[(\lambda+\alpha+\beta)\lambda(\lambda-\lambda z+\mu_1+\mu_2) + \\ (\lambda+\alpha+\beta)\lambda z(-\lambda) - \lambda^2\mu_1 - \beta\lambda\mu_2)]\end{bmatrix}$$

At $z = 1$

$$P(1) = \frac{D(1)N'(1) - D'(1)N(1)}{[D(1)]^2} P_{3,k}$$

$$N(1) = -(\lambda+\theta)\mu_1(\lambda+\alpha)\,\mu_2 + \alpha\,\overline{\gamma}^1\lambda\theta(\mu_2+\beta)(\lambda-\mu_1+1)$$

$$\begin{aligned} N'(1) = {} & -(\lambda+\theta)\mu_1\,[\mu_2\lambda + (\lambda+\alpha)\lambda + (\lambda+\alpha)\,\mu_2 k] - \alpha\,\overline{\gamma}^1\lambda[\beta(\lambda-\mu_1+1)\lambda \\ & + \theta(\lambda-\mu_1+1)\lambda - \theta(\mu_2+\beta) - (\lambda+1) - \theta(\mu_2+\beta)(\lambda-\mu_1+1)k] \end{aligned}$$

$$D(1) = [\theta\lambda\beta\mu_1 + \lambda(\lambda+\alpha)\mu_2 - \alpha\mu_1\mu_2]$$

$$D'(1) = \begin{bmatrix} (-\lambda)[\lambda\beta\mu_1 + \lambda(\lambda + \alpha)\,\mu_2 - \alpha\mu_1\mu_2] + \\ +\theta\lambda[(\alpha + \beta)\mu_1 + (\lambda + \alpha)\mu_2 - \lambda(\lambda + \alpha + \beta)] \end{bmatrix}$$

6. Numerical Results:

The curved graph constructed in Figure 2 and the values tabulated in the Table 1 are obtained by setting the fixed values $\mu_2 = 1$, $\alpha = 1$, $\beta = 0.3$, $k = 1$, $\theta = 1$, $\gamma = 0.1$, $\sigma = 0.1$, $t = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of μ_1 from 1 to 1.2 in steps of 0.1. We observed that as λ rises L_s also rises which shows the stability of the model.

The curved graph constructed in Figure 3 and the values tabulated in the Table2 are obtained by setting the fixed values $\mu_1 = 1$, $\mu_2 = 1$, $\beta = 0.3$, $k = 1$, $\theta = 1$, $\gamma = 0.1$, $\sigma = 0.1$, $t = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of α from 1.2 to 1.8 in steps of 0.3. We observed that as λ rises L_s also rises which shows the stability of the model.

The curved graph constructed in Figure 4 and the values tabulated in theTable 3 are obtained by setting the fixed values $\mu_1 = 1$, $\mu_2 = 1$, $\alpha = 1$, $k = 1$, $\theta = 1$, $\gamma = 0.1$, $\sigma = 0.1$, $t = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of β from 3 to 9 in steps of 3. We observed that as λ rises L_s also rises which shows the stability of the model.

The curved graph constructed in Figure 5 and the values tabulated in theTable4 are obtained by setting the fixed values $\mu_1 = 1$, $\alpha = 1$, $\beta = 0.3$, $k = 1$, $\theta = 1$, $\gamma = 0.1$, $\sigma = 0.1$, $t = 0.3$ and varying the values of λ from 1 to 2 incremented with 0.2 and extending the values of μ_2 from 0.5 to 1.5 in steps of 0.5. We observed thatas λ rises L_s also rises which shows the stability of the model.

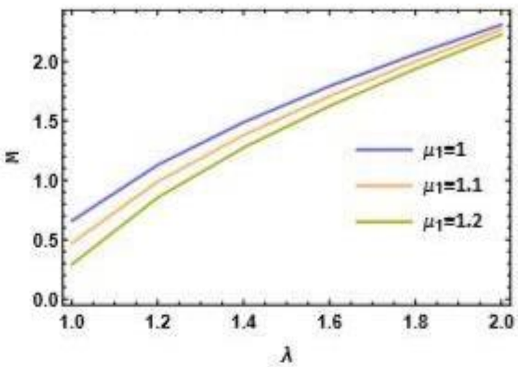

Figure 2: $E(L)$ with turn over of μ_1

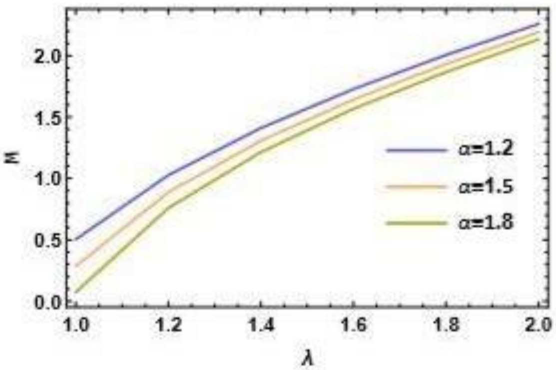

Figure 3: $E(L)$ with turn over of α

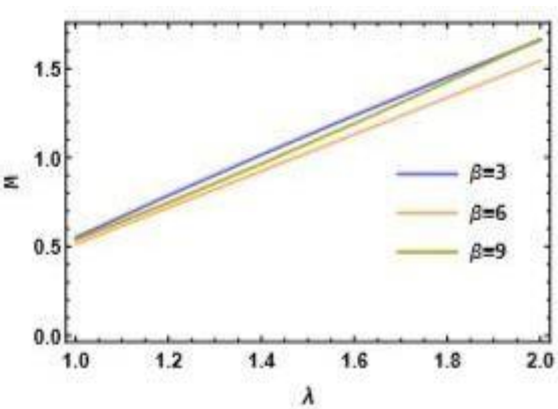

Figure 4: $E(L)$ with turn over of β

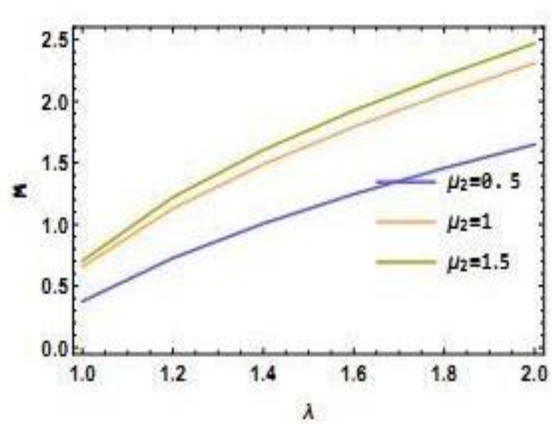

Figure 5: $E(L)$ with turn over of μ_2

Table 1: Ls with turn over λ

$\mu_1=1$	$\mu_1=1.1$	$\mu_1=1.2$
0.6578	0.4714	0.2920
1.1290	0.9892	0.8550
1.4885	1.3798	1.2748
1.7915	1.7078	1.6255
2.0609	1.9999	1.9373
2.3085	2.2691	2.2248

Table 2: Ls with turn over λ

α =1.2	α =1.5	α =1.8
0.5060	0.2870	2.1924
1.0281	0.8886	0.7056
1.4119	1.3080	0.7608
1.7285	1.6438	1.2150
2.0063	1.9329	1.5687
2.2590	2.1924	2.1333

Table 2: Ls with turn over λ

β =3	β =6	β =9
0.5060	0.2870	2.1924
0.78819	0.7192	0.7473
1.0161	0.9246	0.9651
1.2364	1.1306	1.1913
1.4507	1.3369	1.4253
1.6598	1.5432	1.6664

Table 3: Ls with turn over λ

μ_2=0.5	μ_2=1	μ_2=1.5
0.3761	0.6578	0.7044
0.7269	1.290	1.2212
1.0074	1.4885	1.6047
1.2460	1.7915	1.9248
1.4572	1.7915	1.9248
1.6491	2.3085	2.4702

7. Conclusion

In this paper, analysis of a Markovian retrial queue with recurrent customers, switch over time and server vacation is evaluated. We obtain the PGF for the number of customers and the mean number of customers in the orbit. We work out the waiting time distribution. We also derive the performance measures. We perform some particular cases. We illustrate some numerical results.

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