η-RICCI SOLITON ON LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLD

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Abstract
The objective of this paper is to study some properties of n-dimensional η-Ricci Lorentzian concircular structure manifold (briefly $(LCS)_n$ manifold) whose metric is Ricci soliton. The conditions on η-Ricci and η-Ricci flat $(LCS)_n$ manifolds have been obtained for being shrinking, steady and expanding and derived the parallel conditions. The conditions have been discussed on concircular vector field equipped with η-Ricci soliton to be steady and concircular flat η-Ricci soliton also have been discussed. It is proved that sectional curvature of any plane section in manifold is constant. It is also proved that η-Ricci semisymmetric manifold is Einstein manifold.

Keywords: η-Ricci Soliton, η-Ricci $(LCS)_n$ manifold, η-Ricci flat $(LCS)_n$ manifold, η-Einstein manifold.

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DOI: - 10.48047/ecb/2022.11.10.25

1. Introduction

A.K. Mandal [11] studied Ricci soliton and gradient Ricci soliton in an LP-Sasakian manifold in 2014. In 2015, A.M. Blaga [4, 5] introduced the generalized form of Ricci soliton i.e., η-Ricci soliton on Para-Sasakian manifolds and Lorentzian Para-Sasakian manifold. Some properties of $(LCS)_n$ manifold was studied by Hui, Chakraborty [10], S.K. Yadav, D.L. Suthar and M. Hailu worked on extended generalized phi-recurrent $(LCS)_{2n+1}$ manifold [15]. In 2014, B.Y. Chen and S. Deshmukh [7] worked on the geometry of compact shrinking Ricci soliton and derived some conditions to be shrinking. Bejan and Crasmareanu [3] studied Ricci solitons with quasi constant curvature.

In 2013, Ashoka, Bagewadi and Ingala~~ha~~lli derived certain results on Ricci solitons in α-Sasakian manifolds [2] and in 2014 they worked on geometry on Ricci solitons in $(LCS)_n$ manifold. Ingalahalli and Bagewadi [12] developed the concept of Ricci soliton on α-Sasakian manifolds. Later on, Nagaraja and Premalatha [14] studied Ricci soliton on Kenmotsu manifolds. M.M. Tripathi [13] also showed his contribution in the study of Ricci soliton on contact metric manifolds. Hamilton [9] defined the Ricci flow on a Riemannian manifold by $\frac{\partial}{\partial t}\varpi(k) = -2$.

The Ricci soliton on a Riemannian manifold (M, ϖ) is a triplet (ϖ, ξ , λ) satisfying

$$(L_\xi\varpi)(A,B) + 2\gamma(A,B) + 2\lambda\varpi(A,B) = 0, \quad (1.1)$$

where, γ denotes the Ricci tensor, L_ξ is the Lie derivative along the vector field ξ and $\lambda \in \mathbb{R}$ for a Riemannian metric ϖ on M for all vector fields A and B.

The generalization of Ricci soliton that was introduced by Cho and Kimura [12] is called an η-Ricci soliton and defined by

$$(L_\xi\varpi)(A,B) + 2\gamma(A,B) + 2\lambda\varpi(A,B) +$$

$2\mu\eta(A)\eta(B) = 0.$ (1.2)

The η-Ricci soliton reduces to Ricci soliton for $\mu = 0$.

2. Preliminaries

Definition 2.1 A $(LCS)_n$ manifold is said to be an Einstein manifold if its Ricci tensor γ is of the form $\gamma(A,B) = a\varpi(A,B)$, (2.1)
for non-vanishing constant a. The Einstein manifolds with $a = 0$ are called Ricci flat manifolds.

Definition 2.2 A $(LCS)_n$ manifold is said to be an η-Einstein manifold if its Ricci tensor γ is of the form
$\gamma(A,B) = a\varpi(A,B) + b\eta(A)\eta(B)$, (2.2)
where a and b are non-vanishing constants.

Definition 2.3 A vector field A is called concircular vector field if it satisfies the relation
$(\nabla_A\eta)\,B = \alpha\{\varpi(A,B) + \eta(A)\eta(B)\}$, (2.3)
where α is a non- zero scalar function and η is a 1- form.

If ξ is taken as a concircular characteristics vector field then we have
$\nabla_A\xi = \alpha\{A + \eta(A)\xi\}$, (2.4)
for all vector fields A, B such that α satisfy
$\nabla_A\alpha = A\alpha = d\alpha = \rho\eta(A)$. (2.5)
ρ is a scalar function given by $\rho = -\xi\alpha$.

In Lorentzian concircular structure manifold briefly $(LCS)_n$ manifold (M^n, ϖ) (n>2), we have the followingrelation:

$\eta(\xi) = -1$, $\phi\xi = 0$, and $\varpi(\phi A, \phi B) = \varpi(A,B) + \eta(A)\eta$. (2.6)

$\phi^2 A = A + \eta(A)\xi$. (2.7)

$\gamma(A,\xi) = (n-1)(\alpha^2 - \rho)\eta(A)$. (2.8)

$R(A,B)\xi = (\alpha^2 - \rho)[\eta(B)A - \eta(A)B]$. (2.9)

$R(\xi,B)W = (\alpha^2 - \rho)[\varpi(B,W)\xi - \eta(W)B]$. (2.10)

$(\nabla A\phi)\,B = \{R(A,B) + 2\eta(A)\eta(B)\xi + \eta(B)A\}$. (2.11)

$R(A,B)W = \phi R(A,B)W + (\alpha^2 - \rho)\{\varpi(B,W)\eta(A) - \varpi(A,W)\eta(B)\}\xi$. (2.12)

$\gamma(\phi A, \phi B) = \gamma(A,B) + (n-1)(\alpha^2 - \rho)\eta(A)\eta(B)$. (2.13)

For any vector field B, W, where R and γ denote the curvature tensor and Ricci tensor respectively.
The Lie derivative is given by
$(L_\xi\varpi)(A,B) = \varpi(\nabla_A\xi, B) + \varpi(A, \nabla_B\xi)$. (2.14)

3. η- Ricci soliton on $(LCS)_n$ manifold

Using (2.4) in (2.14), we get
$\frac{1}{2}(L_\xi\varpi)(A,B) = \alpha[\varpi(A,B) + \eta(A)\eta(B)$. (3.1)
Using (3.1) in (1.2), we get
$\gamma(A,B) = -(\alpha+\lambda)\varpi(A,B) - (\alpha+\mu)\eta(A)\eta(B)$. (3.2)
From (3.2), we can state

Theorem 3.1 *An η-Ricci soliton equipped with $(LCS)_n$ manifold is an η-Einstein manifold.*

By putting $B = \xi$ in (3.2), we get
$\gamma(A,\xi) = (\mu - \lambda)\eta(A)$. (3.3)
Contracting (3.2), we get
$r = -(\alpha+\lambda)n + (\alpha+\mu)$. (3.4)

Corollary 3.2 *An η-Ricci soliton equipped with $(LCS)_n$ manifold satisfy the relation (3.3) and (3.4).*

Theorem 3.3 *If the concircular vector field equipped with $(LCS)_n$ manifold is geodesic, then Ricci soliton is steady.*

Proof: Using (3.2) in (1.2), we get

$$\varpi(\nabla_A\xi, B) + \varpi(A, \nabla_B\xi) - 2(\alpha - \lambda)\varpi(A, B) - 2(\alpha + \mu - \mu)\eta(A)\eta(B) = 0. \quad (3.5)$$

Taking $A = B = \xi$ in (3.5), we get

$$(\varpi(\nabla\xi\xi, \xi) + \varpi(\xi, \nabla\xi\xi) - 2(\alpha - \lambda)\varpi(\xi, \xi) - 2\alpha\eta(\xi)\eta(\xi) = 0. \quad (3.6)$$

$$2(\nabla\xi\xi, \xi) + 2(\alpha - \lambda) - 2\alpha = 0. \quad (3.7)$$

$$(\nabla\xi\xi, \xi) = \lambda. \quad (3.8)$$

Since $\nabla\xi\xi = 0$ which implies $\lambda = 0$ and hence the result is proved.

4. Concircular curvature tensor in *η*-Ricci soliton

In an $(LCS)_n$ manifold, the concircular curvature tensor $\mathcal{L}$ is defined by

$$\mathcal{L}(A,B)W) = R(A,B)W - \frac{r}{n(n-1)}[\varpi(B,W)A - \varpi(A,W)B]. \quad (4.1)$$

Theorem 4.1: *For concircular flat η–Ricci soliton with $(LCS)_n$ manifold is*

- *Expanding if $\mu > \frac{r}{n}$,*
- *Steady if $\mu = \frac{r}{n}$ and*
- *Shrinking if $\mu < \frac{r}{n}$.*

Proof: In view of equation (4.1), we get

$$\mathcal{L}(A,B,W,D) = R(A,B,W,D) - \frac{r}{n(n-1)}[\varpi(B,W)\varpi(A,D) - \varpi(A,W)\varpi(B,D)], \quad (4.2)$$

where $\mathcal{L}(A,B,W,D) = \varpi(\mathcal{L}(A,B)W,D)$ and $A,B,W,D \in \chi(M)$.

For vanishing concircular curvature tensor, equation (4.2) yields

$$R(A,B,W,D) = \frac{r}{n(n-1)}[\varpi(B,W)\varpi(A,D) - \varpi(A,W)\varpi(B,D)], \quad (4.3)$$

Contracting (4.3) by taking $A = D = e_i$, we get

$$\gamma(A,B) = \frac{r}{n(n-1)}[\varpi(B,W)n - \varpi(e_i,W)\varpi(B,e_i)]. \quad (4.4)$$

$$\gamma(A,B) = \frac{r}{n(n-1)}[\varpi(B,W)n - \varpi(B,W)]. \quad (4.5)$$

Comparing (4.5) and (3.2), we get

$$\left[-\alpha - \lambda - \frac{r}{n}\right]\varpi(B,W) = [\alpha + \mu]\eta(W)\eta(B). \quad (4.6)$$

Taking $B = W = \xi$ in (4.6), we get

$$\lambda = \mu - \frac{r}{n}. \quad (4.7)$$

Therefore, for concircular curvature tensor, η–Ricci soliton with $(LCS)_n$ manifold is

- *Expanding if $\mu > \frac{r}{n}$,*
- *Steady if $\mu = \frac{r}{n}$ and*
- *Shrinking if $\mu < \frac{r}{n}$.*

Theorem 4.2: *In an η-Ricci flat $(LCS)_n$ manifold, if $\mu \neq 0$, then the soliton is non- steady.*

Proof: Using definition (2.2) in (1.2), we get

$$\varpi(\nabla_A\xi, B) + \varpi(A, \nabla_B\xi) + 2\varpi(A,B) + 2\mu\eta(A)\eta(B) = 0. \quad (4.8)$$

$$(\alpha + \lambda)\varpi(A,B) + (\alpha + \mu)\eta(A)\eta(B) = 0. \quad (4.9)$$

Taking $A = B = \xi$ in (4.9), we get

$$\lambda = \mu. \quad (4.10)$$

If $\mu \neq 0$, then η-Ricci flat $(LCS)_n$ manifold is non-steady.

Theorem 4.3: In an η-Ricci flat Einstein manifold, the Ricci tensor Υ is of the form

$$\gamma(A,B) = \frac{3r - (n-1)(\alpha^2 - p) - 4\alpha n}{n}\varpi(A,B) + (2r - 4\alpha n - (n-1)(\alpha^2 - p))\eta(A)\eta(B).$$

Proof: In $(LCS)_n$ manifold,

$$\gamma(A,B) = [(n-1)(\alpha^2 - p) - r + 2\alpha n]\varpi(A,B) + (r - 2\alpha n)\eta(A)\eta(B). \quad (4.11)$$

On contracting (2.3), we get

$$r = na - b. \quad (4.12)$$

Taking $A = B = \xi$ in (2.3) and comparing the result with (4.11), we get

$$b - a = 2r - 4\alpha n - (n-1)(\alpha^2 - p).$$

(4.13)

Solving (4.12) and (4.13) for a and b, we get

$$a = \frac{3r-(n-1)(\alpha^2-p)-4\alpha n}{n} \quad \text{and}$$

$$b = 2r - 4\alpha n - (n-1)(\alpha^2 - p).$$

Hence, the result is proved.

Theorem 4.4: *In $(LCS)_n$ manifold with metric structure $(\phi, \xi, \eta, \varpi)$, the sectional curvature of any plane section is equal to $-\alpha^2$.*

Proof: Using Riemannian curvature tensor, we have

$$R(\xi, A)\xi = \nabla_\xi \nabla_A \xi - \nabla_A \nabla_\xi \xi - \nabla_{[\xi,A]}\xi.$$

$$R(\xi, A)\xi = \alpha \nabla_\xi(\phi A) - \alpha\phi(\nabla_\xi A) - \alpha\phi(\nabla_A \xi),$$

which implies

$$R(\xi, A)\xi = -\alpha^2 \phi^2 A.$$

(4.14)

Using (2.7) in (4.14), we get

$$R(\xi, A)\xi = -\alpha^2(A + \varpi(A, \xi)\xi),$$

If A is orthogonal to ξ, then $R(\xi, A)\xi = -\alpha^2 A$.

$$\varpi(R(\xi, A)\xi, A) = -\alpha^2.$$

Hence the result is proved.

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