Complex Symmetric Operators and Numerical Ranges in Fuzzy Semi-Inner Product Spaces

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Abstract
On a fuzzy semi-inner product (FSIP) space, we propose the numerical ranges and complex symmetric bounded linear operators. We determine the complex symmetric operator and the numerical ranges and also complex symmetry in the fuzzy dual space of FSIP is established. In fuzzy semi-inner product space, we demonstrate some features of a generalized adjoint of a complex symmetric operator and significant numerical ranges of operators in this space.
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1. Introduction

The concept of semi-inner product (SIP) space was first presented by Lumer [1]. Giles [2] shows that it is feasible to create a SIP with a few desirable internal product characteristics in a sizable class of Banach spaces. Takagi [3] investigated the $n \times n$ symmetric complex matrix $Ta = \lambda\bar{a}$, where $\bar{a}$ stands for the conjugation of the vector a in $\mathbb{C}^n$. Chō and Tanahashi [4] defined a conjugation on a complex Banach space and also explored some spectral characteristics of complex symmetric operators. The set of complex numbers with the form $\langle T\xi, \xi \rangle$ with ξ ranging across the unit vectors in H, is known as the numerical range of $T \in \mathcal{L}(\mathcal{H})$. The numerical range has numerous uses and is extremely helpful for understanding operators [5]. The numerical ranges of antilinear operators and conjugations operating on a Hilbert space were also researched by Hur and Lee [6]. Recently, numerical ranges in semi-inner product spaces were introduced by An and Heo [7].

On the other hand, Zadeh [8] is the one who first introduces the idea of fuzzy sets. Subsequently, Nanda [9] and Matloka [10] studied fuzzy number sequences and introduced the l_p^f, l_∞^f fuzzy number sequence spaces. The notion of the fuzzy norm on a linear space was initially developed by Katsaras [11] while researching fuzzy topological vector spaces. There has been some research done in fuzzy inner product spaces, which is a study that is still in its early stages. Among the first to provide a comprehensive characterization of fuzzy inner product space and the accompanying fuzzy norm functions were ElAbyad and Hamouly [12] and Biswas [13]. Goudarzi and Vaezpour [14], Kohli and Kumar [15], and Majumder and Samanta [16] individually introduced the fuzzy inner product that Mukherjee and Bag [17] modified. The concepts of fuzzy normed algebra and fuzzy points were introduced by Ramakrishnan [18]. Fuzzy points are used to define FSIP.

The numerical range of a bounded linear operator on an FSIP space is examined in this research. We determine the numerical ranges of numerous operators using the general FSIP on $\ell_n^\rho(\mathbb{C})$, with the ℓ^ρ-norm $(1 \leq \rho < \infty)$. In an FSIP space, we also develop a conjugation and a complex symmetric operator and explore their fundamental characteristics. We also address the numerical ranges of operators multiplied by conjugations on FSIP spaces that are sequentially essential.

2. Preliminaries

Definition 2.1 [1] Let R be a complex vector space. A function $[\cdot,\cdot]: R \times R \to \mathbb{C}$ is a SIP on R if the following properties are satisfied: for any $a, b, c \in R$,
(1) $[a+b, c] = [a, c] + [b, c]$,
(2) $[\lambda a, b] = \lambda[a, b]$ for all $\lambda \in R$,
(3) $[a, a] > 0$ for $a \neq 0$,
(4) $|[a, b]|^2 \leq [a, a][b, b]$.

Definition 2.2. [19] A FSIP on a unitary $M(I)$-modulo R is a function $::R \times R \to M(I)$ holds true under the three conditions below;
(F_1) $\cdot$ is linear in one component only.
(F_2) $a \cdot a > \bar{0}$ for every nonzero $a \in$ R.
$(F_3)|[a \cdot b]|^2 \leq (a \cdot a)(b \cdot b)$ for every $a, b \in R$.

Definition 2.3. [20] On a complex or real vector space, a fuzzy pseudo-norm R is a function $\|: a \to R^*(I)$ that is satisfied for the field variables $a, b \in R$ and t.
(i) $\| ta \| = |t| \| a \|$
(ii) $\| a + b \| \leqslant \| a \| \oplus \| b \|$.
Such $\|$ is referred to as a fuzzy norm if it also satisfies
(iii) For each nonzero $a \in R$, $\|a| > \bar{0}$.

Definition 2.4. [21] If R is a set, then a_λ is considered to be a fuzzy point on R,
$$R_\lambda(a) = \begin{cases} \lambda & \text{where } \lambda \in (0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5. [18] Let R be a fuzzy normed algebra $C(I)$ module and the set of all fuzzy points on R is $\hat{R}$. An FSIP of fuzzy points on R is a function $*$:
$\hat{R}\ x\ \hat{R} \to C(I)$ if the following conditions are satisfied:
(i) * is linear in the first argument, i.e. $(a_\lambda \oplus b_\mu) * c_\psi = a_\lambda * c_\psi + b_\mu * c_\psi$ and $(\alpha a_\lambda) * c_\psi = \alpha(a_\lambda * c_\psi), \forall \alpha \in C(I)$.
(ii) $a_\lambda * a_\lambda > \bar{0}\ \forall\ a_\lambda \neq 0_\beta$ where $\beta \in (0,1]$.
(iii) $[\,[a_\lambda * b_\mu]\,]^2 \leqslant [\,[a_\lambda * a_\lambda]\,][\,[b_\mu * b_\mu]\,]$ where a_λ, b_μ and $c_\psi \in \hat{a}$. So, $\langle \hat{a}, * \rangle$ is referred to as an FSIP space of fuzzy points.

3. Numerical ranges of FSIP space operators

The numerical range in a finite-dimensional FSIP space is explicitly computed in this section. We identify the $\mathbb{C}^n$ complex n-dimensional space that is fitted with the ℓ^p-norm as $\ell_n^p(\mathbb{C})(p \geq 1)$.

Definition 3.1. Let $\langle \hat{R}, *, \|\|\rangle$ be an FSIP space of fuzzy points. Let T be a fuzzy bounded linear map on $\langle \hat{R}, *, \|\|\rangle$. If $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ acts on $\ell_2^\rho(\mathbb{C})(1 \leq \rho < \infty)$, then the fuzzy numerical range $W(T)$ is the [0,1] closed interval since $[Tu * u]_\rho = |a|^\rho$ for any unit vector $u = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$W(T) = \{Tu_\rho * u_\rho \mid \|u_\rho\| = \bar{1}\},$$

$$[\,[W(T)]\,] = \mathrm{Sup}_{u_\rho \in \hat{R}}\{[\,[Tu_\rho * u_\rho]\,] \mid \|u_\rho\| = \bar{1}\}.$$

Example 3.2. Let $\langle \hat{R}, *, \|\|\rangle$ be an FSIP space of fuzzy points. Let T be a fuzzy bounded linear map on $\langle \hat{R}, *, \|\|\rangle$. For $1 < \rho < \infty$, let the closed disc be C_ρ with radius $\frac{1}{\rho}(\rho-1)^{\frac{\rho-1}{\rho}}$ centred at the origin. For $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $\ell_2^\rho(\mathbb{C})$, clearly, fuzzy numerical range $W(T)$ is the closed disc C_ρ of FSIP space.

To prove that $W(T) = C_p$, such that any unit vector $u = \begin{pmatrix} a \\ b \end{pmatrix}$ in $\ell_2^\rho(\mathbb{C})$ with $|a|^\rho + |b|^\rho = 1$. Since $[Tu * u]_\rho = b\bar{a}|a|^{\rho-2}$ and $|b|^\rho = 1 - |a|^\rho$, we have

$$\left|[Tu * u]_\rho\right|^\rho = \mathrm{Sup}_{u_\rho \in \hat{R}}\{\|Tu_\rho\|(1 - |a|^\rho) \mid \| \, |a|^{\rho(\rho-1)} \, \| = \bar{1}\}.$$

Since the function $f(t) = (1 - t^\rho)t^{\rho(\rho-1)}$ has the maximum value $\frac{1}{\rho^\rho}(\rho-1)^{\rho-1}$ at $t = \left(\frac{\rho-1}{\rho}\right)^{1/\rho}$, $\left|[Tu * u]_\rho\right|$ has the maximum value $\frac{1}{\rho}(\rho-1)^{\frac{\rho-1}{\rho}}$ when $|a|^\rho = \frac{\rho-1}{\rho}$. This demonstrates that the FSIP disc C_ρ is closed over $W(T)$.

Theorem 3.3. Let $\langle \hat{R}, *, \|\|\rangle$ be an FSIP space of fuzzy points. Let T be a fuzzy bounded linear map on $\langle \hat{R}, *, \|\|\rangle$. Let T represent the reverse shift on an infinite-dimensional Banach space, $\ell^\rho(\mathbb{C})$, for $1 \leq \rho < \infty$. The open unit disc is then the fuzzy numerical range $W(T)$.

Proof Let $a = (a_1, a_2, a_3, \dots)$ be any unit vector in $\ell^\rho(\mathbb{C})$, and let $k = \min\{i \geq 1 : a_i \neq 0\}$. Then we have

$$\left|[Ta * a]_\rho\right| \leq \sum_{j=k}^{\infty} \|Ta_\rho * a_\rho\| |a_{j+1}| |a_j|^{\rho-1}$$

$$\leq \|Ta_\rho\| \frac{1}{\rho} \sum_{j=k}^{\infty} \{|a_{j+1}|^\rho + (\rho-1)|a_j|^\rho\}$$

$$= \|Ta_\rho\| \frac{1}{\rho} \left\{(\rho-1)|a_k|^\rho + \rho \sum_{j=k+1}^{\infty} |a_j|^\rho\right\}$$

$$= \|Ta_\rho\| \frac{1}{\rho} \left\{\rho \sum_{j=k}^{\infty} |a_j|^\rho - |a_k|^\rho\right\} = 1 - \frac{|a_k|^\rho}{\rho} \leq \bar{1}.$$

As a result, we get that$[\,[W([Ta * a]_\rho)]\,] \leqslant \left|[Ta * a]_\rho\right| < 1$, for any unit vector $a \in \ell^\rho(\mathbb{C})$, which implies that the open unit disc contains $W(T)$.

In the open unit disc, let $f(\omega) = re^{i\theta}$ be any vector with $0 \leq r < 1$ to demonstrate the reverse inclusion. We take the vector $f(a) \in \ell^{\rho}(\mathbb{C})$ given by

$$f(a) = \left((1-r^{\rho})^{\frac{1}{\rho}}, r(1-r^{\rho})^{\frac{1}{\rho}}e^{i\theta}, r^2(1-r^{\rho})^{\frac{1}{\rho}}e^{2i\theta}, r^3(1-r^{\rho})^{\frac{1}{\rho}}e^{3i\theta}, \ldots\right).$$

Then we see that $[\,[f(a_{\rho})]\,] \leqslant re^{i\theta}(1-r^{\rho})\|a_{\rho}\|, \forall a_{\rho} \in \hat{R}$.
$[\,[f(a_{\rho})]\,] = \| a \|_{\rho} = 1$, so that $[a,a]_{\rho} = \| a \|_{\rho}^2 = 1$. Moreover, we get that

$$[Ta * a]_{\rho} = re^{i\theta}(1-r^{\rho})\|Ta_{\rho} * a_{\rho}\| \sum_{k=1}^{\infty} r^{(k-1)\rho}$$

$$= re^{i\theta}(1-r^{\rho})\|Ta_{\rho}\| \sum_{k=1}^{\infty} r^{(k-1)\rho} re^{i\theta} = \mathrm{f}(\omega),$$

that implies the open unit disc is contained in $W(T)$. Hence the proof.

Corollary 3.4. Let $T = \binom{xy}{0}$ act on ℓ_2^1, where $x, z \in \mathbb{R} \setminus \{0\}, x+z \neq 0$, and $y \in \mathbb{C}$. For every unit vector $u = (a * b)^t \in \ell_2^1$, and let $u = (a * b)^t \in \ell_2^1$. If y is also a nonzero fuzzy real integer, then for $a \neq 0$ and b, where $|\mathrm{f}(b)| = 1 - |\mathrm{f}(a)|$,

$$W(T) \subset \left\{ f(\omega) \in \mathbb{C}: |\mathrm{Re}(\mathrm{f}(\omega))| \leq \frac{|\mathrm{f}(x+y+z)|}{2} \text{ and } |\mathrm{Im}(\mathrm{f}(\omega))| \leq \frac{|\mathrm{f}(y)|}{2} \right\}$$

Proof. If $f(a) = a_1 + ia_2$ and $f(b) = b_1 + ib_2$, where f_{a_j} and f_{b_j} are fuzzy real numbers ($j = 1,2$), then

$$[Tu * u]_1 = \left\{ \mathrm{f}(x-z)|a| + \mathrm{f}(z) + \frac{f(y)}{|a|}(a_1b_1 + a_2b_2) \right\} + i\frac{f(y)}{|a|}(a_1b_2 - a_2b_1).$$

We obtain that for $t := \sqrt{a_1^2 + a_2^2}$, by using the Cauchy-Schwarz inequality.

$$\begin{aligned} f(a_1b_1 + a_2b_2)^2 &\leq \left(f_{a_1^2} + f_{a_2^2}\right) * \left(f_{b_1^2} + f_{b_2^2}\right) \\ &= (a_1^2 + a_2^2)\left(1 - \sqrt{a_1^2 + a_2^2}\right)^2 \\ &= t^2(1-t)^2 =: f(t). \end{aligned}$$

On the fuzzy interval $(0,1)$, f has the maximum 0.06 at $t = 0.5$, so we get $|a_1b_1 + a_2b_2| \leq 0.25$, and this gives the inequality

$$|\mathrm{Re}([Tu * u]_1)| \leq \mathrm{Sup}_{u \in \hat{R}} \left\{ [Tu * u]| \frac{|\mathrm{f}(x+y+z)|}{2} \leq \bar{1} \right\}$$

We also get the inequality in this instance,
$|\mathrm{Im}([Tu * u]_1)| = \mathrm{Sup}_{u \in \hat{R}}\{[Tu * u]|\,|2y(a_1b_2 - a_2b_1)|\} \leq \frac{|\mathrm{f}(y)|}{2} \leq \bar{1}$ by a similar method.
Hence the proof.

4. Complex symmetric operators and conjugations on FSIP space

In this section, we will employ an FSIP to describe a conjugation on an FSIP space. Unless otherwise stated, R in this section indicates an FSIP space with an FSIP $[\cdot,\cdot]$.

Definition 4.1. Let $\langle \hat{R}, *, \|\|\rangle$ be an FSIP space of fuzzy points. An operator $C: R \to R$ is a conjugation if it is involutive ($C^2 = I_R$) and

$$[Ca_{\rho} * Cb_{\rho}] = \overline{[a_{\rho} * b_{\rho}]} = \|a_{\rho}\|\,\|b_{\rho}\|, \text{ for all } a_{\rho}, b_{\rho} \in R.$$

Proposition 4.2. Let $\langle \hat{R}, *, \|\| \rangle$ be an FSIP space of fuzzy points. Assume relation exists if C is a conjugation of R,

$$C^2 = I_R,\ \| C \| \leq 1,\ C(a_\rho + b_\rho) = Ca_\rho + Cb_\rho \text{ and } C(\omega a_\rho) = \bar{\omega} C a_\rho$$

holds for all $a_\rho, b_\rho \in R$ and $\omega \in \mathbb{C}$.

Proof. Clearly from the Cauchy-Schwarz inequality for an FSIP, $\| Ca_\rho \|^2 = [Ca_\rho * Ca_\rho] = \overline{[a_\rho * a_\rho]} \leq \| a_\rho \|^2$ for each $a_\rho \in R$, that exists $\| C \| \leq 1$. Since

$$\begin{aligned}[C(a_\rho + b_\rho) * Cc_\rho] &= \overline{[a_\rho + b_\rho * c_\rho]} = \overline{[a_\rho * c_\rho]} + \overline{[b_\rho * c_\rho]} \\ &= [Ca_\rho * Cc_\rho] + [Cb_\rho * Cc_\rho] = [Ca_\rho + Cb_\rho * Cc_\rho] = \|Ca_\rho + Cb_\rho\| \, \|Cc_\rho\|\end{aligned}$$

for all $a_\rho, b_\rho, c_\rho \in R$. Considering that the operator C is surjective, we can select $c_\rho \in R$ such that

$$Cc_\rho := C(a_\rho + b_\rho) - Ca_\rho - Cb_\rho$$

Then we get that $0 = [C(a_\rho + b_\rho) - Ca_\rho - Cb_\rho * Cc_\rho] = [Cc_\rho * Cc_\rho] = \|c_\rho\| \, \|c_\rho\|$, so that $Cc_\rho = 0$, that is, $C(a_\rho + b_\rho) = Ca_\rho + Cb_\rho$. To prove that $C(\omega a_\rho) = \bar{\omega} C a_\rho$ for any $a_\rho \in R$ and $\omega \in \mathbb{C}$, chosose any element $\omega \in R$ there exists

$$\begin{aligned}[C(\omega a_\rho) * b_\rho] &= \overline{[\omega a_\rho * Cb_\rho]} = \overline{\omega [a_\rho * Cb_\rho]} \\ &= \bar{\omega}[Ca_\rho * b_\rho] = [\bar{\omega} C a_\rho * b_\rho] = [\bar{\omega} \|a_\rho\| \, \|b_\rho\|].\end{aligned}$$

This corresponds to $C(\omega a_\rho) = \bar{\omega} C a_\rho$. C thus satisfies the fuzzy relation.

Definition 4.3. Let $\langle \hat{R}, *, \|\| \rangle$ be an FSIP space of fuzzy points. Let T be a fuzzy bounded linear map on $\langle \hat{R}, *, \|\| \rangle$. Suppose C is the conjugation of R. If $T \in \mathcal{L}(R)$ is C-symmetric, then for all $a_\rho, b_\rho \in R$,

$$\begin{aligned}[Ta_\rho * b_\rho] &= [a_\rho * CTCb_\rho] \\ &= \operatorname{Sup}_{a_\rho \in \hat{R}} \{ [\, [Ta_\rho * b_\rho] \,] \mid \|a_\rho\| = \bar{1} \} .\end{aligned}$$

Proposition 4.4. Let T be a fuzzy bounded linear map on $\langle \hat{R}, *, \|\| \rangle$. Let C be a conjugation on a uniform FSIP space R.

(i) If $T \in \mathcal{L}(R)$ is C-symmetric, then $T^* \in \mathcal{L}(R^*)$ is also C^*-symmetric.

(ii) If $T \in \mathcal{L}(R)$ is C-symmetric, then $(T^\perp)^* = (T^*)^\perp$.

(iii) If $\{T_n\}$ is a sequence of C-symmetric operators there exists $T_n \to S$ in the strong fuzzy topology,
then S is C-symmetric.

Proof. (i) Assume T is an operator on R that is C-symmetric. Let's assume that p and q are free-form entities in the fuzzy dual space $R^\star$. There are particular vectors a_ρ and a_ρ in R that have the properties ${a_\rho}^* = p$ and ${a_\rho}^* = q$ because R is a uniform FSIP space. We note that $T^* {a_\rho}^* = (T^\perp a_\rho)^*$. Indeed, for any $c_\rho \in R$, we have

$$(T^* * {a_\rho}^*)(c_\rho) = {a_\rho}^*(T * c_\rho) = [Tc_\rho * a_\rho] = [c_\rho * T^\perp a_\rho] = (T^\perp * a_\rho)^*(c_\rho).$$

Moreover, for any $c_\rho \in R$ and ${b_\rho}^* \in R^\star$, we see that

$$(C^* T^* C^*){b_\rho}^*(c_\rho) = {b_\rho}^*(CTCc_\rho) = (CTC)^\star {b_\rho}^*(c_\rho)$$

so $C^* T^* C^* = (CTC)^*$. Thus we have

$$\begin{aligned}\left[T^{*}a_{\rho}{}^{*} * b_{\rho}{}^{*}\right]_{\star} &= \left[b_{\rho} * T^{\perp}a_{\rho}\right] = \left[CT^{\perp}Cb_{\rho} * a_{\rho}\right] \\ &= \left[a_{\rho}{}^{\star} * \left(CT^{\perp}Cb_{\rho}\right)^{\star}\right]_{\star} = \left[a_{\rho}{}^{\star} * (CTC)^{\star}b_{\rho}{}^{*}\right]_{\star} \\ &= \left[a_{\rho}{}^{\star} * C^{*}T^{*}C^{\star}b_{\rho}{}^{*}\right]_{*}\end{aligned}$$

That represents T^{*} is C^{*}-symmetric.

(ii) For every $c_{\rho} \in R$ and $b_{\rho}{}^{*} \in R^{*}$, we get

$$(T^{\perp})^{*}b_{\rho}{}^{*}(c_{\rho}) = (CTC)^{*}b_{\rho}{}^{*}(c_{\rho}) = (C^{\star}T^{*}C^{*})b_{\rho}{}^{*}(c_{\rho})$$

Yet, it is evident from (i) that T^{*} is C^{*}-symmetric. Therefore, for every $a_{\rho}{}^{*} \in R^{*}$,

$$\left[a_{\rho}{}^{*} * (T^{*})^{\perp}b_{\rho}{}^{*}\right]_{\star} = \left[T^{*}a_{\rho}{}^{*} * b_{\rho}{}^{*}\right]_{*} = \left[a_{\rho}{}^{*} * (C^{*}T^{*}C^{*})a_{\rho}{}^{*}\right]_{\star}.$$

This means that $(T^{\perp})^{\star} = (T^{*})^{\perp}$.

(iii) Since $\|(S - T_{n})a_{\rho}\| \to 0$ for all $a_{\rho} \in R$, for all $a_{\rho}, b_{\rho} \in R$, we have

$$\left[CSCa_{\rho} * b_{\rho}\right] = \lim_{n\to\infty}\left[CT_{n}Ca_{\rho} * b_{\rho}\right] = \lim_{n\to\infty}\left[a_{\rho} * T_{n}b_{\rho}\right] = \left[a_{\rho} * Sb_{\rho}\right] = S\|a_{\rho}\|\,\|b_{\rho}\|.$$

where uniform continuity leads to the third equality. S is a C-symmetric operator as a result.

We now calculate a conjugation's fuzzy numerical range on $\ell_{n}^{\rho}(\mathbb{C})$.

Theorem 4.5. Let $\langle \hat{R}, *, \|\|\rangle$ be an FSIP space of fuzzy points. Let C be a complex conjugation on $\ell_{n}^{\rho}(\mathbb{C})(1 \le \rho < \infty)$ given by $Ca_{\rho} = \bar{a_{\rho}} = \left(\bar{a_{\rho}}_{1}, \dots, \bar{a_{\rho}}_{n}\right)$ for $a_{\rho} \in \ell_{n}^{\rho}(\mathbb{C})$. Then

(1) $W(C) = \{\omega \in \mathbb{C}: |\omega| = 1\}$ for $n = 1$,

(2) $W(C) = \{\omega \in \mathbb{C}: |\omega| \le 1\}$ for $n \ge 0$.

The proof is trivial for (i). For any $a_{\rho} \in \ell_{1}^{\rho}(\mathbb{C})$ with $|a_{\rho}| = 1$, we write $a_{\rho} = e^{i\theta}$ for some real number θ. We have $[Ca * a]_{\rho} = [\bar{a} * a]_{\rho} = e^{-2i\theta}$, and so $W(C) = \{\omega \in \mathbb{C}: |\omega| = 1\}$. To prove (ii), let $a_{\rho} \in \ell_{n}^{\rho}(\mathbb{C})$ be any unit vector, i.e., $\| a \|_{\rho}^{2} = [a * a]_{\rho} = 1$. As a result of the Cauchy-Schwarz inequality, we obtain

$$\left|[Ca * a]_{\rho}\right|^{2} \le [Ca * Ca]_{\rho}[a * a]_{\rho} = \overline{[a * a]_{\rho}}[a * a]_{\rho} = \|a_{\rho}\|\,\|a_{\rho}\|=1$$

that exists $W(C) \subseteq \{\omega \in \mathbb{C}: |\omega| \le 1\}$.

Let ω be any complex number with $|\omega| \le 1$ for the reverse inclusion. For some real number θ, a polar form is represented as $f(\omega) = |\omega|e^{i\theta}$. Now consider the unit vector $a \in \ell_{n}^{\rho}(\mathbb{C})$ given by

$$a_{\rho} = \left(\left(\frac{1 + |\omega|}{2}\right)^{\frac{1}{\rho}} e^{-\frac{i\theta}{2}}, \left(\frac{1 - |\omega|}{2}\right)^{\frac{1}{\rho}} ie^{-\frac{i\theta}{2}}, 0, \dots, 0\right).$$

Then we have

$$[Ca * a]_{\rho} = [\bar{a} * a]_{\rho} = \|a_{\rho}\|\,\|a_{\rho}\|,\ = \left(\frac{1 + |\omega|}{2} - \frac{1 - |\omega|}{2}\right)e^{i\theta} = |\omega|e^{i\theta} = \mathrm{f}(\omega),$$

this suggests that the closed unit disc is represented in $W(C)$. The closed unit disc is thus the fuzzy numerical range $W(C)$.

Assume that C is the typical complex conjugation. Such that

$w(C) = \sup\{\left|[Ca * a]_{\rho}\right|: \left[a_{\rho} * a_{\rho}\right] = \|a_{\rho}\|\,\|a_{\rho}\|,\ = 1, a_{\rho} \in \ell_{n}^{\rho}\} = 1$ for all $n \ge 1$, where the fuzzy numerical radius of C is denoted by $w(C)$. Moreover, we can identify an infinite number of unit vectors that satisfy the complex conjugation C on $\ell_n^1(\mathbb{C})(n \geq 1)$ the fuzzy numerical radius that is, vectors a_ρ with $\left|[Ca_\rho * a_\rho]_1\right| = \|a_\rho\| \, \|a_\rho\|, \; = 1$.

Theorem 4.6. Let T be a fuzzy bounded linear map on $\langle \hat{R}, *, \|\|\rangle$. Let $T \in \mathcal{L}(R)$, and let C be a conjugation on R. Such that $W(CTC) = \overline{W(T)}$ and $W_s(T) = \overline{W_s(CTC)}$ that $\bar{A}$ represents the complex conjugation of A.

Proof. If $c_\rho \in W(CTC)$, then a vector $a_\rho \in R$ exists with $[a_\rho * a_\rho] = 1$ such that

$$c = [CTCa_\rho * a_\rho] = CT\|a_\rho\| \, \|a_\rho\|, \; = \overline{[TCa_\rho * Ca_\rho]} \in \overline{W(T)}$$

the fuzzy numerical range of T on an FSIP space R by

$$W_s(T) = \Big\{c_n \in \mathbb{C}: \lim_n [Ta_n * a_n| \, \|a_n\|] = c \text{ for any } \{a_n\} \subset R \text{ with } [a_n * a_n] = \|a_n\| \, \|a_n\|, \\ = \bar{1}, a_n \xrightarrow{w} 0\Big\}$$

$[\,[W_s(T)]\,] = \mathrm{Sup}_{a_n \in \hat{R}}\{[\,[Ta_n * a_n]\,] \mid \|a_n\| = \bar{1}\}$.

Such that $W(CTC) \subset \overline{W(T)}$. Since $W(T) = W(C^2TC^2) \subset \overline{W(CTC)}$, we get the reverse inclusion. Hence, we have $W(CTC) = \overline{W(T)}$.

If $c \in W_s(CTC)$, then a sequence $\{a_n\} \subset R$ exists with $[a_n * a_n] = \|a_n\| \, \|a_n\| = \bar{1}$ and $a_n \xrightarrow{w} 0$. Since $\lim_n a_n = 0$ in the weak sense, clearly $\lim_n f(a_n) = 0$ for all $f \in R^*$. Since $C^* f \in R^\star$ for all $f \in R^\star$, we have $\lim_n f(Ca_n) = \lim_n C^\star f(a_n) = 0$, which implies that $Ca_n \xrightarrow{w} 0$. Thus we have

$$c = \lim_n [CTCa_n * a_n] = \; CT\|a_n\| \, \|a_n\|, \; = \lim_n \overline{[TCa_n * Ca_n]} \in \overline{W_s(T)}$$

which implies that $W_s(CTC) \subset \overline{W_s(T)}$. Following the reverse inclusion is

$$W_s(T) = W_s(C^2TC^2) \subset \overline{W_s(CTC)}$$

Hence the proof.

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