A NOTE ON DOMSATURATION NUMBER AND DOMSATURATION POLYNOMIAL OF A GRAPH

W. Jinesha[1,*], D. Nidha[2]

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Abstract

Let G be a simple graph of order n. The domsaturation polynomial of a graph G of order n is the polynomial $Ds(G,x) = \sum_{i=ds}^{n} d(G,i)\, x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. The domsaturation number of G is the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. In this paper, we obtain the domination polynomial and minimal domination polynomial of a graph. For any positive integer $m \geq 1$ and $p \geq 2$, there exists a domsaturation polynomial such that $P(x) = \sum_{i=1}^{mp} \{\sum_{j=0}^{m} mC_j [(m-j)pC_{i-j(p-1)}]\}\, x^{i+m}$. We also characterize certain graphs for which $ds(G)$ is of class 1 and class 2. For any tree T with $n \geq 2$, there exists a vertex $v \in V$ such that $ds(T-v) = ds(T)$. Also, we study the domination polynomial and roots for a zero-divisor graph.

Keywords: domination polynomial, domsaturation number, domsaturation polynomial, zero-divisor graph.

Mathematical Classification: 05C31, 05C69

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[1]Research Scholar [Reg.No:20213112092012], [2]Assistant Professor [1,2]Research Department of Mathematics, Nesamony Memorial Christian College, Marthandam - 629 165, Tamil Nadu, India.

[1,2]Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India.

*[*]**Coressponding author:*** [1]wsjineshaa@gmail.com , [2]dnmaths@nmcc.ac.in

Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex $V \setminus S$ is adjacent to at least one vertex in S. The domination number γ in G is the minimum cardinality of a dominating set in G. Fundamentals of domination and several advanced topics are given in Haynes et.al. A dominating set with cardinality $\gamma(G)$ is called a γ-set. An i-subset of $V(G)$ is a subset of $V(G)$ of cardinality i. Let $\mathcal{D}(G, i)$ be the family of dominating sets of G which are i-subsets and let $d(G, i) = |\mathcal{D}(G, i)|$. The polynomial $D(G, x) = \sum_{i=\gamma}^{|V(G)|} d(G, i)x^i$ is defined as domination polynomial of G. A root of $D(G, x)$ is called a domination root of G. For any vertex u of G, the eccentricity of u is $e(u) = max\{d(u, v);\ v \in V\}$. The diameter $diam\ G$ is defined as $diam\ G = max\{e(v); v \in V\}$. Acharya introduced the concept of domsaturation number ds of a graph. The least positive integer k such that every vertex of G lies in a dominating set of cardinality k is called the domsaturation number of G and is denoted by $ds(G)$.

Definition 1.1. [7] A graph G is said to be of class 1 or class 2 according as $ds = \gamma$ or $\gamma + 1$.

Definition 1.2. [9] A tree T of order 3 or more is a caterpillar if the removal of its leaves produces a path.

Notation 1.3. [2] A graph obtained by joining any number of isolated vertices to each pendant vertex of a graph G is denoted by $G(S)$.

Notation 1.4. [2] If G is a graph with vertex set $V = \{u_1, u_2, \dots\}$, then the graph obtained by identifying one of the end vertices of n_2 copies of P_2, n_3 copies of P_3,... at u_1, m_2 copies of P_2, m_3 copies of P_3,... at u_2,..` is denoted by $G[u_1(n_2P_2, n_3P_3, \dots); u_2(m_2P_2, m_3P_3, \dots);\ \dots]$.

Theorem 1.5. [2] If G is a tree, then $\gamma(G) = 2$ if and only if G is either $P_2(S)$ or $P_3(S)$ or $P_4(S)$.

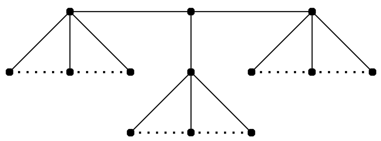
Theorem 1.6. [6] Let $n \geq 2$ be a natural number. The size of the smallest dominating set containing both end vertices of P_n is $\left\lceil \frac{n+2}{3} \right\rceil$. Moreover, if $n \geq 4$ and $\not\equiv 1(mod\ 3)$, there are at least two dominating sets of size $\left\lceil \frac{n+2}{3} \right\rceil$ containing both end vertices of P_n.

Theorem 1.7. [6] Let G be a connected graph with exactly two distinct domination roots. Then $(G, x) = x^n (x + 2)^n$, where n is a natural number. Indeed $G = H \circ K_1$, for some graph H of order n.

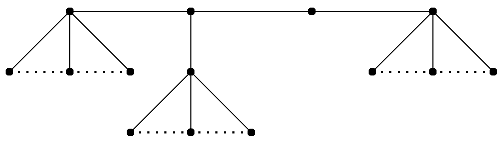
Theorem 1.8. [7] Let T be a caterpillar. Then T is of class 1 if and only if every support is adjacent to exactly one pendant vertex and for any two consecutive supports u and v, $d(u, v) \equiv 1(mod\ 3)$.

Theorem 1.9. [7] The path P_n of order n is of class 1 if and only if $n \equiv 1(mod\ 3)$.

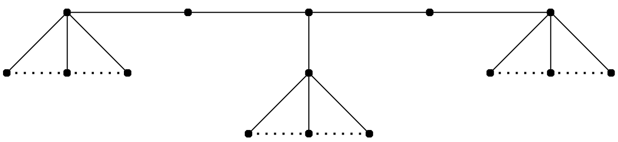
Theorem 1.10. [2] If G is a tree, then $\gamma(G) = 3$ if and only if G is either $P_3[u_1(k_1P_2); u_2(k_2P_2); u_3(k_3P_2)]$ or $P_4[u_1(k_1P_2); u_2(k_2P_2); u_4(k_3P_2)]$ or $P_5[u_1(k_1P_2);\ u_5(k_2P_2)]$ or $P_5[u_1(k_1P_2); u_2(k_2P_2)); u_5(k_3P_2)]$ or $P_5[u_1(k_1P_2); u_3(k_2P_2); u_5(k_3P_2)]$ or $P_6[u_1(k_1P_2); u_6(k_2P_2)]$ or $P_6[u_1(k_1P_2); u_3(k_2P_2); u_6(k_3P_2)]$ or $P_7[u_1(k_1P_2); u_7(k_2P_2)]$ or $P_7[u_1(k_1P_2); u_4(k_2P_2); u_7(k_3P_2)]$ or any one of the graphs given in the below figure



T_1



T_2



T_3

Figure 1: Trees satisfying $\gamma(G) = 3$

Main Result

Theorem 2.1. For any positive integer $m \geq 1$ and $p \geq 2$, there exists a graph having polynomial such that

$$P(x) = \sum_{i=0}^{mp}\{\sum_{j=0}^{m} mC_j[(m-j)pC_{i-j(p-1)}]\}\ x^{\,i+m} \quad (1)$$

Proof. Let $P = (v_1, v_2, \ldots, v_m)$ be a path on m vertices. Attach the pendant vertices $u_1, u_2, \ldots, u_p$, $(p \geq 2)$ to each v_i, $i = 1,2,\ldots,m$. For the resulting graph, we have $\gamma(G) = m$, which is minimal and the γ-set is unique. Since the γ-set is unique, the co-efficient of x^m is 1.
Case(i). $m = 1, p \geq 2$. Therefore $\gamma(G) = 1$. Now, we find the co-efficient of x^{m+i}. For $i = 1$, $\{v_1\} \cup \{u_a\}$, $1 \leq a \leq p$ is a dominating set of cardinality $m+1$. There are pC_1 choices. Therefore the number of dominating sets of cardinality 2 is pC_1. Proceeding like this, for $i = mp - 1$, there are pC_{p-1} choices. Also we can remove the support vertex v_1 and add all the pendant vertices attached to that v_1. In this case, the number of dominating sets of cardinality p is $pC_{p-1} + 1$. For $i = mp$, the one and only one choice is to choose all the vertices. Therefore $P(x) = x + pC_1x^2 + pC_2x^3 + \ldots + [pC_{p-1} + 1]x^p + x^{p+1}$. For $m = 1$, (1) equation contains only the first two terms and the remaining terms gets eliminated, because $mCj\ [(m-j)pC_{i-j(p-1)}] = 1Cj\ [(1\ -\ j)pC_{i-j(p-1)}] = 0, j\ =\ 2,3,\ldots,m; i = 0,1,\ldots,mp$. Therefore, (1) reduces to

$$P(x) = \sum_{i=0}^{p}\{pC_i + 1C_1[0C_{i-(p-1)}]\}\,x^{\,i+1}, p \geq 2$$

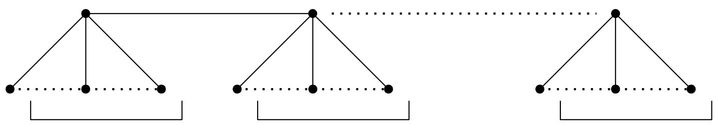
Now, $0C_{i-(p-1)} = 1$, for $i = p - 1$. Therefore

$$P(x) = \sum_{i=0}^{mp}\{\sum_{j=0}^{m} mC_j[(m-j)pC_{i-j(p-1)}]\}\ x^{\,i+m}.$$

Case(ii). $m > 1$ and $p \geq 2$.

Consider the dominating set of cardinality $m + 1$.

Subcase(i). $p = 2$. Here $\{v_1, v_2, \ldots, v_m\} \cup \{u_a\}$, $1 \leq a \leq mp$ is a dominating set of cardinality $m + 1$. In this case, the are mpC_1 choices. Since $p = 2$, $\{v_1, v_2, \ldots, v_m\} - \{v_i\} \cup \{u_a, u_b\}$, where u_a and u_b are the pendant vertices of same v_i, $1 \leq i \leq m$, $1 \leq a \leq mp$ and $1 \leq b \leq mp$. Therefore there are mC_1 choices. Hence the number of dominating sets of cardinality $m + 1$ is $mpC_1 + m$ for $p = 2$.



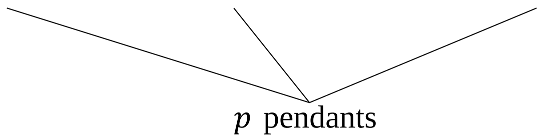


Figure 2: Graph respresenting $m > 1$ and $p \geq 2$

Subcase(ii). $p > 2$. The only way of getting the dominating set of cardinality $m + 1$ is $\{\{v_1, v_2, \ldots, v_m\} \cup \{u_a\},\ 1 \leq a \leq mp$ for $p > 2$. Otherwise at least one non-dominated pendant vertex will be found.

Therefore the number of dominating sets of cardinality $m + 1$ is $\begin{cases} mpC_1 + mC_1 & for\ p = 2 \\ mpC_1 & for\ p > 2 \end{cases}$.

For $i = 1, m > 1$ and $p \geq 2$, the co-efficient of x^{m+1} in (1) reduces to

$$mpC_1 + mC_1[(m - 1)pC_{1-(p-1)}] \qquad (2)$$

The remaining terms will become zero, because $i - j(p - 1) < 0$, for $j \geq 2$.

Therefore (2) becomes $\begin{cases} mpC_1 + mC_1 & for\ p = 2 \\ mpC_1 & for\ p > 2 \end{cases}$.

Now, we consider the dominating set of cardinality $m + 2$.

Subcase(i). p=3. Here $\{v_1, v_2, \ldots, v_m\} \cup \{u_a, u_b\}$, $a \neq b$, $1 \leq a, b \leq mp$ is a dominating set of cardinality $m + 2$. Therefore there are mpC_2 choices. Since $p = 3$, we remove any one of the support vertex and add pendant vertices attached to that v_i , otherwise at least one pendant vertex of v_i is not dominated. $\{v_1, v_2, \ldots, v_m\} - \{v_i\} \cup \{u_a, u_b, u_c\}$, $1 \leq i \leq m$, $a \neq b \neq c$, and $1 \leq a, b, c \leq mp$ is also a dominating set of cardinality $m + 2$. There are mC_1 choices. Therefore we get $mpC_2 + mC_1$ dominating sets of cardinality $m + 2$ for $p = 3$.

Subcase(ii). $p > 3$. The only way to choose the dominating sets of cardinality $m + 2$ is $\{v_1, v_2, \ldots, v_m\} \cup \{u_a, u_b\}$, $a \neq b$, $1 \leq a, b \leq mp$, for $p > 3$. Otherwise at least one non-dominated pendant vertex will be found. Therefore the number of dominating sets of cardinality $m + 2$ is mpC_2.

Subcase(iii). $p < 3$, that is $p = 2$. Clearly $\{v_1, v_2, \ldots, v_m\} \cup \{u_a, u_b\}$ is a dominating set of cardinality m + 2. Then we remove any one of the support vertex v_i and add two pendant vertices which are attached to that v_i and one pendant vertex from different v_i , $i = 1,2, \ldots, m$. In this case, we can choose $mC_1[(m - 1)pC_1]$ ways. It is also possible to remove two support vertices and add four pendant vertices, that is $\{v_1, v_2, \ldots, v_m\} - \{v_i, v_j\} \cup \{u_a, u_b, u_c, u_d\}$, $i \neq j$, $a \neq b \neq c \neq d$, $1 \leq i, j \leq m$, $1 \leq a, b, c, d \leq mp$, where u_a and u_b are attached to either v_i or v_j and u_c and u_d are attached to either v_i or v_j. Otherwise at least one non-dominated pendant vertex will be found. In this case there are mC_2 choices. Therefore the number of dominating sets of cardnality $m + 2$ is $mpC_2 + mC_1[(m - 1)pC_1] + mC_2$. From the above cases, we get

The number of dominating sets of cardinality

$$m+2=\begin{cases} mpC_2+mC_1 & for\ p=3 \\ mpC_2 & for\ p>3 \\ mpC_2+mC_1[(m-1)pC_1]+mC_2 & for\ p<3 \end{cases}.$$

For $i=2$, $m>1$ and $p\geq 3$, the coefficient of x^{m+2} in (1) reduces to

$$mpC_2+mC_1[(m-1)pC_{2-(p-1)}]+mC_2[(m-2)pC_{2-2(p-1)}] \qquad (3)$$

The remaining terms will become zero because $2-j(p-1)<0$, for $j>3$ and $p-1>3$.

Therefore (3) becomes $m+2=\begin{cases} mpC_2+mC_1 & for\ p=3 \\ mpC_2 & for\ p>3 \\ mpC_2+mC_1[(m-1)pC_1]+mC_2 & for\ p<3 \end{cases}.$

Continuing this way, now we consider the dominating set of cardinality mp. Here $i=m(p-1)$. Suppose all the vertices belongs to the dominating set and we add $m(p-1)$ pendant vertices. There are $mpC_{m(p-1)})$ choices. Suppose, if we remove one support vertex then we must add all the pendant vertices which are attached to that support vertex and $(m-1)(p-1)$ pendants from the remaining support vertices. In this case, there are $mC_1[(m-1)pC_{(m-1)(p-1)}]$ choices. Continuing the above process, it is possible to remove all the support vertices and add all the pendant vertices to get the dominating set of cardinality mp. In this case there is only one choice. Therefore the number of dominating sets of cardinality mp is

$$mpC_{m(p-1)})+mC_1[(m-1)pC_{(m-1)(p-1)}]+mC_2[(m-2)pC_{(m-2)(p-1)}]+\ldots+1.$$

For $i=m(p-1)$, $m\geq 1$ and $p\geq 3$, the co-efficient of x^{mp} in (1) becomes

$$\left\{\sum_{j=0}^{m} mC_j\left[(m-j)pC_{i-j(p-i)}\right]\right\}x^{mp}=\left\{\begin{matrix} mpC_{m(p-1)}+mC_1\left[(m-1)pC_{(m-1)(p-1)}\right] \\ +\ldots+1 \end{matrix}\right\}x^{mp}.$$

Consider the dominating set of cardinality $>mp$. Suppose all the support vertices belong to the dominating set then we have to add greater than $m(p-1)$ pendants to get a dominating set of cardinality $>mp$. There are $mpC_{m(p-1)+j}$ choices, $j=1,2,\ldots,m$. We can remove one support and add the pendants attached to that support also we add greater than $(m-1)(p-1)$ pendants fom the remaining supports to get a dominating set of cardinality $>mp$. In this case, there are $mC_1[(m-1)pC_{(m-1)(p-1)+j}]$ choices, $j=1,2,\ldots,m-1$. Continuing in this way, we can remove $m-1$ supports and add the pendants attached to that support also we add $p-1$ pendants from the remaining support. In this case, there are $mC_{m-1}[pC_{\ (p-1)+j}]$ choices, $j=$

1, that is m choices. Combining all the cases ,we get

$P(x) = \sum_{i=0}^{mp} \{\sum_{j=0}^{m} mC_j [(m-j)pC_{i-j(p-1)}]\} \ x^{\ i+m}$.

■

Definition 2.2. The domsaturation polynomial of a graph G of order n is the polynomial $Ds(G,x) = \sum_{i=ds}^{n} d(G,i)\, x^i$, where $d(G,i)$ is the number of dominating sets of G of size i and ds is the domsaturation number of G.

Theorem 2.3. For any positive integer $m \geq 1$ and $p \geq 2$, there exists a domsaturation polynomial such that

$$P(x) = \sum_{i=1}^{mp} \left\{ \sum_{j=0}^{m} mC_j [(m-j)pC_{i-j(p-1)}] \right\} x^{\ i+m}.$$

Proof. Let $P = v_1 v_2 \dots v_m$ be a path on m vertices. Attach the pendant vertices $u_1, u_2, \dots,\ u_p$, $(p \geq 2)$ to each v_i, $i = 1,2, \dots, m$. For the resulting graph, the support vertices belongs to the dominating set but the pendant vertices does not lie in a dominating set of cardinality γ. Therefore $ds = m + 1$. The remaining part follows from the above theorem. ■

Theorem 2.4. Let T be a caterpillar graph, such that each vertex in P_n has atleast one pendant vertex then

1. $ds + \overline{ds} = m + 2$ if T is of class 1.
2. $ds + \overline{ds} = m + 3$ if T is of class 2.

Proof. Let m be the number of support vertices, then the cardinality of the minimal dominating set will be m, that is $\gamma = m$.

1. Suppose T is of class 1, then $ds = \gamma = m$. For T, $\overline{ds} = 2$. Therefore, $ds + \overline{ds} = m + 2$.

2. Suppose T is of class 2 $ds = \gamma + 1 = m + 1$. Therefore, $ds + \overline{ds} = m + 3$. ■

Theorem 2.5. Let $n \geq\ 2$ be a natural number.Then the domsaturation number of P_n containing both end vertices of P_n is $\left\lceil \frac{n+4}{3} \right\rceil$.

Proof. We apply induction on n. For $n = 2,3,4$ the assertion is trivial. Assume that $x \in V(P_n)$, $deg(x) = 2$, $N(x) = \{y,z\}$ and $deg(y) = deg(z) = 2$. Let $N(y) = \{x,a\}$ and $N(z) = \{x,b\}$. Remove x,y,z and join a and b. Let k be the domsaturation number of P_n. Let γ_k-set be a dominating set of cardinality k. By induction hypothesis, the size of the smallest dominating set of cardinality k containing end vertices of P_{n-3} is $\left\lceil \frac{n+1}{3} \right\rceil$. If DS is a γ_k-set for P_{n-3} of size $\left\lceil \frac{n+1}{3} \right\rceil$ containing end vertices and $a \in DS$, then $DS \cup \{z\}$ is a γ_k-set for P_n of size

$\left\lceil\frac{n+4}{3}\right\rceil$. If DS is a γ_k-set for P_{n-3} of size $\left\lceil\frac{n+1}{3}\right\rceil$ containing end vertices and $b \in DS$, then $DS \cup \{y\}$ is a γ_k-set for P_n of size $\left\lceil\frac{n+4}{3}\right\rceil$. If $a, b \notin DS$, then $DS \cup \{x\}$ is a γ_k-set for P_n of size $\left\lceil\frac{n+4}{3}\right\rceil$. Now, suppose that P_n has a γ_k-set, say DS of size less than $\left\lceil\frac{n+4}{3}\right\rceil$ containing both end vertices of P_n. It is not hard to see that there exists a $v \in DS$ such that $deg(v) = 2$ and $N(v) \cap DS = \emptyset$. Consider $DS \setminus N[v]$ and join two pendant vertices of two components $DS \setminus N[v]$ to obtain a path of order $n-3$. This path has a γ_k-set of size less than $\left\lceil\frac{n+1}{3}\right\rceil$ containing end vertices of P_{n-3}, a contradiction. ■

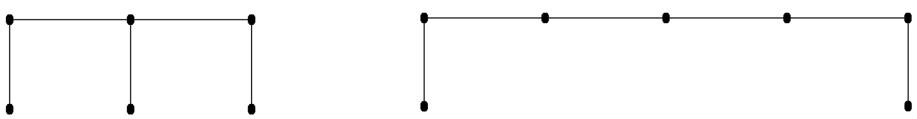
Theorem 2.6. For any tree T with $n \geq 2$, there exists a vertex $v \in V$ such that $ds(T-v) = ds(T)$.

Proof. Clearly, the result is true if $T = K_2$. Assume T has at least one vertex v, with $deg(v) \geq 2$ that is adjacent to at least one end vertex and atmost one non end vertex. If v is adjacent to two or more end vertices u_1 and u_2, then v is in every γ-set for T but the pendants does not belong to any γ-set. In this case $ds(T-u_1) = ds(T)$. If not, then v is adjacent to one end vertex u and $deg(v) = 2$. Let $T' = T - v - u$. For any graph G, if $deg(u) = 1$, then $ds(G - u) \leq ds(G)$. Hence $ds(T') \leq ds(T-u) \leq ds(T)$. However, $ds(T') \geq ds(T) - 1$. If $ds(T') = ds(T) - 1$, then $ds(T) = ds(T-v)$. Otherwise, $ds(T') = ds(T) = ds(T-u)$. ■

Proposition 2.7. If each vertex in the path P_n is attached to the wounded spider by subdividing $t-1$ edges, the $\gamma = ds = \Gamma$.

Proposition 2.8. Let G be a non complete graph of order n, then $\frac{n}{1+\Delta} = ds(G)$ if and only if $\Delta(G) = n-1$.

Theorem 2.9. If G is a tree, then $ds(G) = 3$ if and only if G is either $P_2(S)$ with at least one support has more than one pendant or $P_3(S)$ or $P_4(S)$or any one of the graphs given in the below figure.



T_1 T_2

Figure 3: Trees satisfying $ds(G) = 3$

Proof. We shall prove this theorem by 2 cases.

Case(i). G is of class 1, then $\gamma(G) = ds(G)$. It is enough to show that $\gamma(G) = 3$ and G is of class 1. By theorem 1.8, $G \cong T_1$. Also the path P_7 is of class 1, then by theorem 1.9, $G \cong T_2$. Otherwise, from theorem 1.10, a non-pendant vertex which is adjacent to support vertex does not belong to any γ-set.

Case(ii). G is of class 2, then $ds(G) = \gamma(G) + 1$. By theorem 1.5, $G \cong P_3(S)\ or\ P_3(S)\ or\ P_2(S)$ with at least one support has more than one pendant. ■

Theorem 2.10. If G is a tree, then $ds(G) = 4$ if and only if G is either $P_3[u_1(k_1P_2); u_2(k_2P_2); u_3(k_3P_2)]$, atleast one $k_i \geq 2$, $1 \leq i \leq 3$ or $P_4[u_1(k_1P_2); u_2(k_2P_2); u_4(k_3P_2)]$ or $P_5[u_1(k_1P_2); u_5(k_2P_2)]$ atleast one $k_i \geq 2$, $1 \leq i \leq 2$ or $P_5[u_1(k_1P_2); u_2(k_2P_2); u_5(k_3P_2)]$ or $P_5[u_1(k_1P_2); u_3(k_2P_2); u_5(k_3P_2)]$ or $P_6[u_1(k_1P_2); u_6(k_2P_2)]$ or $P_6[u_1(k_1P_2); u_3(k_2P_2); u_6(k_3P_2)]$ or $P_7[u_1(k_1P_2); u_7(k_2P_2)]$ or $P_7[u_1(k_1P_2); u_4(k_2P_2); u_7(k_3P_2)]$ or any one of the graphs given in the below figure.

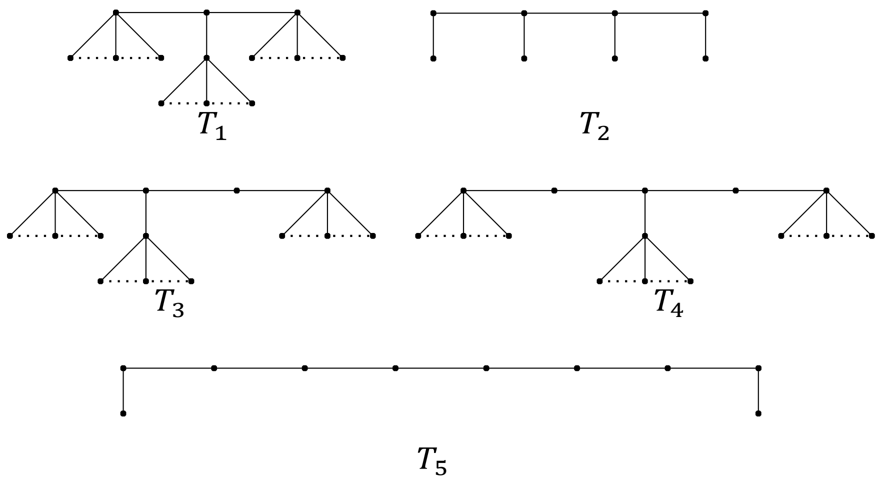


Figure 4: Trees satisfying $ds(G) = 4$

Proof. The proof follows from the above theorem. ■

Problem 2.11. Characterize graphs for which $ds = 5$.

2 Minimal Dominating Polynomial

Definition 3.1. A graph $G = (V\ (G), E(G))$ has $V(G)$ as the vertex set and $E(G)$ as the edge set. A subset $S \subset V\ (G)$ is a dominating set of G, if every vertex in $V(G) - S$ is adjacent to some vertex in S. S is said to be a minimal dominating set if $S - \{u\}$ is not a dominating set for any $u \in S$.

Definition 3.2. The minimal dominating polynomial of a graph G of order n is the polynomial $D(G, x) = \sum_{i=\gamma}^{\Gamma} d(G, i)\, x^i$, where $d(G, i)$ is the cardinality of the minimal dominating sets of G of size i and γ is the minimum cardinality of a minimal dominating set and Γ is the maximum cardinality of a minimal dominating set.

Theorem 3.3. For any positive integer $m \geq 1$ and $p = 2$, there exists a graph having minimal dominating polynomial $x^m(x + 1)^m$.

Proof. Let $P = v_1 v_2 \ldots v_m$ be a path on m vertices. Attach 2 copies of K_1 to each $v_1, v_2, \ldots, v_m$. For m =1, the resulting graph is $K_{1,2}$. Since all the support vertices form a minimal dominating set, it is minimum. The minimal dominating set of cardinality m is 1. Now, for the minimal dominating set of cardinality $m + 1$, we can remove one support vertex and add the pendants attached to that support. In this case, there are mC_1 choices. For the minimal dominating set of cardinality $m + 2$, we can remove two supports and add the pendants attached to that support. In this case, there are mC_2 choices. Proceeding like this, the minimal dominating set of cardinality $2m$ is 1(that is all the pendants). Therefore, the minimal dominating polynomial is

$$x^m + mC_1 x^{m+1} + mC_2 x^{m+1} + \cdots + x^{m+m} = x^m(1 + mC_1 x + mC_2 x^2 + \cdots + x^m)$$
$$= x^m(x + 1)^m. \qquad \blacksquare$$

Problem 3.4. Characterize roots for the polynomial $x^m(x^{p-1} + 1)^m$.

3 Domination Polynomial for Zero-Divisor Graph

Definition 4.1. Let R be a commutative ring(with 1) and let $Z(R)$ be its set of zero-divisors. An element $a \in R$ is called a zero-divisor if there exists a non-zero element $b \in R$ such that $a.b = 0$. Let R be a commutative ring with non-zero identity and let $Z(R)$ be its sets of zero-divisors. The zero-divisor graph of R denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) - 0$, the non-zero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$.

Notation 4.2. The roots of the domination polynomial of a zero-divisor graphs is denoted by $Z(D(\Gamma(R), x)$.

Proposition 4.3. For any prime p, $p \geq 3$, there exists a $\Gamma(Z_{2p})$ with polynomial

$P(\Gamma(Z_{2p}),x) = x^p + px^{p-1} + (p-1)C_{p-3}x^{p-2} + (p-1)C_{p-4}x^{p-3} + \dots +$
$(p-1)C_{\frac{p+1}{2}}x^{p-\frac{p-3}{2}} + (p-1)C_{\frac{p-1}{2}}x^{p-\frac{p-1}{2}} + (p-1)C_{\frac{p-3}{2}}x^{p-\frac{p+1}{2}} + \dots + (p-1)C_1x^2 + x.$

Theorem 4.4. Let $\Gamma(R)$ be a connected zero-divisor graph.Then $Z(D(\Gamma(R),x)) = \{0,-2\}$ if and only if $\Gamma(R) \cong \Gamma(Z_9)$.

Proof. Sine 0 is a domination root with multiplicity $\gamma(\Gamma(R))$, for every graph $\Gamma(R)$ and $D(\Gamma(R),x)$ has two distinct roots, we have $D(\Gamma(R),x) = x^i(x+a)^{m-i}$, for some $i \in N$ and $a > 0$, where $m = |V(D(\Gamma(R)|$. By theorem 1.7, we observe that $\Gamma(Z_9)$ is the only zero-divisor graph having the polynomial $x(x+2)$. Therefore $\Gamma(R) \cong \Gamma(Z_9)$. Converse can be easily verified.

■

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