MINIMAL G#rg-OPEN SETS AND MAXIMAL G#rg-CLOSED SETS IN TOPOLOGICAL SPACES

***Holabasayya I. Sankannavar and Jenifer J. Kernel**

Department of Mathematics

KLE's Polytechnic, Bailhongal-591 102, Karnataka-India

SDM College of Engineering and Technology, Dharwad-580 002, Karnataka-India

Email: *sankannavar555@gmail.com and jeniferjk17@gmail.com

Abstract

In this paper we introduce new class of minimal and maximal sets, minimal G#rg-open and maximal G#rg closed sets and established their relationships with some of generalized classes in topological spaces. Also, we extend the study to cover the generalized minimal and maximal continuous maps in topological spaces.

Mathematics Subject Classification 2010: 54A05

Keywords:maximal G#rg-closed sets, minimal G#rg-open sets, minimal G#rg-continuous maps, minimal G#rg-irresolutemaps.

1. Introduction:

In the years 2001 and 2003, F. Nakaoka and N. Oda [1] and [2] introducedand studied minimal open (resp. minimal closed) sets, which are subclasses of open (resp. closed) sets. The complements of minimal open sets and maximal closed sets are called maximal closed sets and minimal open sets respectively.

In this work, we will discuss the new class of open and closed sets viz. minimal G#rg-open sets and maximal G#rg-closed sets in topological spaces. Also, introduce new class of maps called minimal G#rg-continuous maps and maximal G#rg-continuous maps in topological spaces. These are subclasses of G#rg-open (resp. G#rg-closed) sets respectively.

Definition [1] A proper nonempty open subset U of a topological space X is said to be minimal open set, if any open set which is contained in U is $\emptyset$ or U.

Definition[2] A proper nonempty open subset U of a topological space X is said to be maximal open set, if any open set which contains U is X or U.

Definition[4] A proper nonempty closed subset F of a topological space X is said to be minimal closed set, if any closed set which is contained in F is $\emptyset$ or F.

Definition[4] A proper nonempty closed subset F of a topological space X is said to be maximal closed set, if any closed set which contains F is X or F.

Definition Let X and Y be the topological spaces. A map $f : X \rightarrow Y$ is called:

(i) minimal continuous [3] (briefly min-continuous), if $f^{-1}(M)$ is an open set in X for every minimal open set M in Y.

(ii) maximal continuous [3] (briefly max-continuous), if $f^{-1}(M)$ is an open set in X for every maximal open set M in Y.

(iii) minimal irresolute [3] (briefly min-irresolute), if $f^{-1}(M)$ is minimal open set in X for every minimal open set M in Y.

(iv) maximal irresolute [3] (briefly max-irresolute), if $f^{-1}(M)$ is maximal open set in X for every maximal open set M in Y.

(v) minimal-maximal continuous [3] (briefly min-max continuous), if $f^{-1}(M)$ is maximal open set in X for every minimal open set M in Y.

(vi) maximal-minimal continuous [3] (briefly max-min continuous), if $f^{-1}(M)$ is minimal open set in X for every maximal open set M in Y.

2. Minimal G#rg-open sets and maximal G#rg-closed sets:

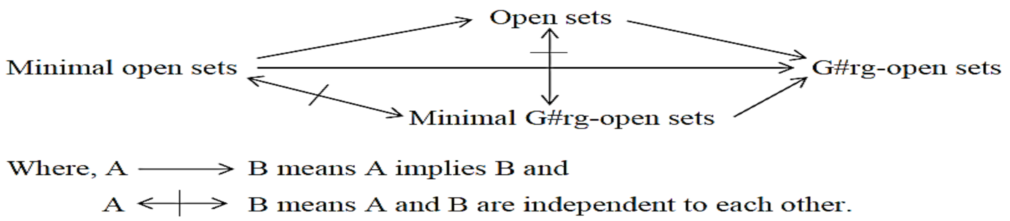
In this section, a new class of sets called minimal G#rg-open sets (maximal G#rg-closed sets) in topological spaces are introduced, which are subclasses of G#rg-open sets (G#rg-closed sets). During this process some of their properties are obtained. Also,we introduce and study maximal G#rg-open sets (minimal G#rg-closed sets) in topological spaces.

1. Definition: Let H be any G#rg-open proper subset of X, H is called minimal G#rg-open set if and only if any G#rg-open set which is contained in H is∅ or H.

2. Remark: Minimal open sets and minimal G#rg-open sets are independent and also open sets and minimal G#rg-open sets are independent each other as illustrated in the following example.

3. Example: Let X = {a, b, c, d} with the topology τ = {X, Ø, {a}, {b, c}, {a, b, c}}, then minimal open sets are {a}, {b, c}. G#rg-open sets in X are X, Ø, {a}, {b}, {c}, {a, b}, {a, c},{b, c}, {a, b, c}. Minimal G#rg-open sets are {a}, {b}, {c}. Here the set {b, c} is a minimal open set (open set) but not a minimal G#rg-open set and the sets {b} and {c} are minimal G#rg-open sets but not minimal open sets (open sets).

4. Remark: From above discussion we have following implications:



5. Theorem: (i) Let G be a minimal G#rg-open set and H be a G#rg-open set,then G ∩ H =∅ or G ⊆ H.

(ii) Let G and H be minimal G#rg-open sets, then G ∩ H =∅ or G = H.

Proof: (i) Let G be a minimal G#rg-open set and H be a G#rg-open set. IfG ∩ H =∅, then there is nothing to prove but if G ∩H ≠ ∅, then we have to prove that G ⊆ H. Suppose G ∩H ≠ ∅. Then G ∩ H ⊆U and G ∩H is G#rg-open, as the finite intersection of G#rg-open sets is a G#rg-open set. Since G is a minimal G#rg-open set, we have G ∩H = G. Therefore G ⊆ H.

(ii) Let G and H be minimal G#rg-open sets. Suppose G ∩ H≠ ∅, then we see that G ⊆ H and H ⊆ G by (i). Therefore G = H.

6. Theorem: Let G be a minimal G#rg-open set. If x is an element of G, then G ⊆H for any open neighbourhood H of x.

Proof: Let G be a minimal G#rg-open set and x be an element of G. Suppose thereexists an open neighbourhood H of x such that G ⊄ H. Then G ∩ H is a G#rg-open set such that G ∩ H ⊆ G and G ∩ H≠ ∅. Since G is a minimal G#rg-open set, wehave G ∩ H= G. That is G ⊆ H. This contradicts our assumption that G ⊄ H. Therefore G ⊆ H for any open neighbourhood H of x.

7. Theorem: Let G be a minimal G#rg-open set. If x is an element of G, then G⊆ H for any G#rg-open set H containing x.

Proof: Let G be a minimal G#rg-open set containing an element x. Suppose there exists a G#rg-open set H containing x such that G ⊄ H. Then G ∩ H is a G#rg-open set such that G ∩H ⊆ G and G ∩ H ≠ ∅. Since G is a minimal G#rg-open set, we have G ∩H = G i.e. G ⊆H. This contradicts our assumption that G ⊄ H. Therefore G ⊆ H for any G#rg-open set H containing x.

8. Theorem: Let G be a minimal G#rg-open set. Then G =∩{H: H is any G#rg-open set containing x} for any element x of G.

Proof: By Theorem 7 and from the fact that G is a G#rg-open set containing x, we have G ⊆∩{H: H is any G#rg-open set containing x}⊆ G. Therefore, we have the result.

9. Theorem: Let G be a nonempty G#rg-open set,then the following three conditions are equivalent.

1) G is a minimal G#rg-open set.
2) G⊆G#rg-cl(A) for any nonempty subset A of G.
3) G#rg-cl(G)=G#rg-cl(A) for any nonempty subset A of G.

Proof: (1)⇒ (2) Let G be a minimal G#rg-open set, x ∈G and A be a nonempty subset of G. By Theorem 7, for any G#rg-open set H containing x, A ⊆G ⊆H which implies A ⊆H. Now A = A∩G ⊆ A ∩ H. Since A is nonempty, therefore A ∩ H≠ ∅. Since H is any G#rg-open set containing x, by Theorem 4.28[6], x∈G#rg-cl(A). That is x∈ G implies x∈G#rg-cl(A) which implies G⊆G#rg-cl(A) for any nonempty subset A of G.

(2) ⇒ (3) Let Abe a nonempty subset of G. That is A⊆G which implies G#rg-cl(A) ⊆G#rg-cl(G)---(i). Again from (2) G⊆G#rg-cl(A) for anynon-empty subset A of G which implies G#rg-cl(G) ⊆G#rg-cl(G#rg-cl(A))= G#rg-cl(A). That is G#rg-cl(G)⊆G#rg-cl(A)--- (ii). From (i) and (ii), we have G#rg-cl(G)=G#rg-cl(A) for any nonempty subset A of G.

(3) ⇒ (1) From (3) we have G#rg-cl(G) = G#rg-cl(A) for any nonempty subsetA of G. Suppose G is not a minimal G#rg-open set. Then there exists anonempty G#rg-open set I such that I⊆G and I≠G. Now there exists anelement a ∈G such that a ∉ I which implies a ∈X −I. That is G#rg-cl({a})⊆G#rg-cl(X −I)=X −I, as X −I is a G#rg-closed set in X. It follows that G#rg-cl({a})≠G#rg-cl(G). This is a contradiction to fact that G#rg-cl({a})=G#rg-cl(G) for anynonempty subset {a} of G. Therefore G is a minimal G#rg-open set.

10. Theorem: Let G be a nonempty finite G#rg-open set. Then there exists at least one (finite) minimal G#rg-open set H such that H⊆G.

Proof: Let G be a nonempty finite G#rg-open set. If G is a minimal G#rg-open set, we may set H = G. If G is not a minimal G#rg-open set, then there exists a (finite) G#rg-open set G_1 such that ∅ ≠G_1⊆G. If G_1 is a minimal G#rg-open set, we may set H = G_1 If G_1 is not a minimal G#rg-open set, then there exists a (finite) G#rg-open set G_2 such that ∅ ≠G_2⊆G_1. Continuing this process, we have a sequence of G#rg-open sets G⊃G_1⊃G_2⊃G_3… ⊃G_k⊃… . Since G is a finite set, this process repeats only finitely. Then finally we get a minimal G#rg-open set H =G_n for some positive integer n.

[A topological space X is said to be locally finite space if each of itselements is contained in a finite open set.]

11. Corollary: Let X be a locally finite space and G be a nonempty G#rg-open set. Then there exists at least one (finite) minimal G#rg-open set Hsuch that H⊆G.

Proof: Let X be a locally finite space and G be a nonempty G#rg-open set. Let x∈G. Since X is a locally finite space, we have a finite open set G_x such that x ∈G_x. Then G∩G_x is a finite G#rg-open set. By Theorem 10 there exists at least one (finite) minimal G#rg-open set H such that H⊆G∩G_x. That is H⊆G∩G_x⊆G. Hence there exists at least one (finite) minimal G#rg-open set H such that H⊆G.

12. Corollary: Let G be a finite minimal open set. Then there exists at least one (finite) minimal G#rg-open set H such that H⊆G.

Proof: Let G be a finite minimal open set. Then G is a nonempty finite G#rg-open set. By Theorem 10, there exists at least one (finite) minimal G#rg-open set H such that H⊆G.

13. Theorem: Let G and G_λ be minimal G#rg-open sets for any element λ of Λ. If G⊆ $\bigcup_{\lambda\in\Lambda} G_\lambda$, then there exists an element λ of Λ such that G = G_λ.

Proof: Let G⊆ $\bigcup_{\lambda\in\Lambda} G_\lambda$. Then G∩($\bigcup_{\lambda\in\Lambda} G_\lambda$)= G. That is $\bigcup_{\lambda\in\Lambda} G \cap G_\lambda$= G. Also by Theorem 5 (ii), $\bigcup_{\lambda\in\Lambda} G \cap G_\lambda$=∅ or G =$G_\lambda$ for any $\lambda \in \Lambda$. It follows that there exists an element $\lambda \in \Lambda$ such that G =G_λ.

14. Theorem: Let G and G_λ be minimal G#rg-open sets for any element λ of Λ. If G $=G_\lambda$ for any element $\lambda \in \Lambda$, then$(\bigcup_{\lambda\in\Lambda} G_\lambda) \cap G = \emptyset$.

Proof: Suppose that $(\bigcup_{\lambda\in\Lambda} G_\lambda) \cap G \neq \emptyset$. That is $\bigcup_{\lambda\in\Lambda}(G_\lambda \cap G) \neq \emptyset$. Then there exits an element $\lambda \in \Lambda$ such that$G_\lambda \cap G \neq \emptyset$. By Theorem 5. (ii), wehave G=G_λ, which contradicts the fact that G $\neq G_\lambda$ for any $\lambda \in \Lambda$. Hence$(\bigcup_{\lambda\in\Lambda} G_\lambda) \cap G = \emptyset$.

15. Theorem: Let G_λ be a minimal G#rg-open set for any element λ of Λ and $G_\lambda \cap G_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$.Assume that $|\Lambda| \geq 2$ Let μ be any element of Λ. Then $(\bigcup_{\lambda\in\Lambda-(M)} G_\lambda) \cap G_\mu = \emptyset$.

Proof: Put G- G_λ in Theorem 14, then we have the result.

16. Corollary: Let G_λ be a minimal G#rg-open set for any element λ of Λ and $G_\lambda \neq G_\mu$ forany elements λ and μ of Λ with $\lambda \neq \mu$. If T is a proper nonempty subset of Λ, then $(\bigcup_{\lambda\in\Lambda-T} G_\lambda) \cap (\bigcup_{\gamma\in T} G_\gamma) = \emptyset$.

17. Theorem: Let G_λ and G_γ be minimal G#rg-open sets for any element $\lambda \in \Lambda$and $\gamma \in$ T. If there exists an element y of T such that $G_\lambda \neq G_\gamma$ for any element λ of Λ, then $(\bigcup_{\gamma\in T} G_\gamma) \not\subset (\bigcup_{\lambda\in\Lambda} G_\lambda)$.

Proof: Suppose that an element γ^1 of T satisfies $G_\lambda \neq G_{\gamma 1}$ for any element $\gamma$of Λ. If $(\bigcup_{\gamma\in T} G_\gamma) \subset (\bigcup_{\lambda\in\Lambda} G_\lambda)$, then we see $G_{\gamma 1} \subset \bigcup_{\lambda\in\Lambda} G_\lambda$. By Theorem 13, there exists an element λ of Λ such that $G_{\gamma 1} = G_\lambda$, which is a contradiction. It follows that $(\bigcup_{\gamma\in T} G_\gamma) \not\subset (\bigcup_{\lambda\in\Lambda} G_\lambda)$.

18. Theorem: Let G_λ be a minimal G#rg-open set for any element λ of Λ and $G_\lambda \neq G_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If T is a proper nonempty subset of Λ, then $(\bigcup_{\gamma\in T} G_\gamma) \subset (\bigcup_{\lambda\in\Lambda} G_\lambda)$.

Proof: Let k be any element of Λ-T. Then $G_k \cap \bigcup_{\gamma\in T} G_\gamma = \bigcup_{\gamma\in T}(G_k \cap G_\gamma) = \emptyset$ and $G_k \cap (\bigcup_{\lambda\in\Lambda} G_\lambda) = \bigcup_{\lambda\in\Lambda}(G_k \cap G_\lambda) = G_k$.If $(\bigcup_{\gamma\in T} G_\gamma) = (\bigcup_{\lambda\in\Lambda} G_\lambda)$, then we have $\emptyset = G_k$. This contradicts our assumption that G_k is a minimal G#rg-open set. Therefore we havetheresult.

19. Definition: Let F be any proper G#rg-closed subset of X is called maximal G#rg-closed set if and only if any G#rg-closed set which contains F is either X or F.

20. Remark: Maximal closed sets and maximal G#rg-closed sets are independent each other as illustrated as following example.

21. Example:Let X = {a, b, c, d} with the topology τ = {X, Ø, {a}, {b, c}, {a, b, c}}, then closed sets in X are X, Ø, {d}, {a, d}, {b, c, d}. Maximal closed sets are {a, d}, {b, c, d}. G#rg-closed sets are X, ∅, {d}, {c, d}, {a, d}, {b, d}, {a, c, d}, {b, c, d}, {a, b, d}, then maximal G#rg-closed sets are {b, c, d}, {a, c, d}, {a, b, d}. But {a, d} is maximal closed set but not maximal G#rg-closed set and also {a, c, d} is maximal G#rg-closed sets but not maximal closed sets in X.

22. Theorem: Let F be any proper subset of X is said to be maximal closed set iff X−F is minimal G#rg-open set in X.

Proof: Let F be a maximal G#rg-closed set. Suppose X−F is not a minimal G#rg-open set. Then there exists aG#rg-open set U ≠X−F such that ∅ ≠U⊂ X−F. That is F ⊂ X−U and X−U is a G#rg-closed set. This contradicts our assumption that F is a minimal G#rg-open set.

Conversely let X−F be a minimal G#rg-open set. Suppose F is not a maximal G#rg-closed set. Then there exists a G#rg-closed set E ≠F such that F ⊂ E ≠ X. That is ∅ ≠ X−E ⊂ X−F and X−E is a G#rg-open set. This contradicts our assumption that X−F is a minimalG#rg-open set. Therefore F is a maximal G#rg-closed set.

23. Theorem: (i) Let F be a maximal G#rg-closed set and E be a G#rg-closed set. Then F ∪ E = X or E⊂ F.

(ii) Let F and E be maximal G#rg-closed sets. Then F ∪ E = X or F = E.

Proof: (i) Let F be a maximal G#rg-closed set and E be a G#rg-closed set. If F ∪ E = X, then there is nothing to prove. But if F ∪ E≠X then we have to prove that E⊂ F. Suppose F ∪ E≠X. Then F ⊂ F ∪ E and F ∪ E is G#rg-closed, as the finite union of G#rg-closed sets is a G#rg-closed set, we have F ∪ E = X or F ∪ E = F. Therefore F ∪ E = F which implies E⊂ F.

(ii) Let F and E be maximal G#rg-closed sets. Suppose F ∪ E≠ X, then we see that F ⊂ E and E⊂ F by (i). Therefore F = E.

24. Theorem: Let F be a maximal G#rg-closed set. If x is an element of F, then for any G#rg-closed set E containing x, F ∪ E = X or E⊂ F.

Proof: Let F be a maximal G#rg-closed set and x is an element of F. Suppose there exists a G#rg-closed set E containing x such that F ∪ E≠ X. Then F ⊂ F ∪E and F ∪ E is a G#rg-closed set, as the finite union of G#rg-closed sets is a G#rg-closed set. Since F is a G#rg-closed set, we have F ∪ E = F. Therefore E⊂ F.

25. Theorem: LetF_α,F_β,F_γ be maximal G#rg-closed sets such that$F_\alpha \neq F_\beta$. If $F_\alpha \cap F_\beta \subset F_\gamma$, then either $F_\alpha = F_\gamma$ or $F_\beta = F_\gamma$.

Proof: Given that $F_\alpha \cap F_\beta \subset F_\gamma$. If $F_\alpha = F_\gamma$ then there is nothing to prove. But if $F_\alpha \neq F_\gamma$ then we have to prove $F_\beta = F_\gamma$. Now $F_\beta \cap F_\gamma = F_\beta \cap (F_\gamma \cap X) = F_\beta \cap (F_\gamma \cap (F_\alpha \cup F_\beta)$ (by Theorem 25) (ii) $= F_\beta \cap ((F_\gamma \cap F_\alpha) \cup (F_\gamma \cap F_\beta))= (F_\beta \cap F_\gamma \cap F_\alpha) \cup (F_\beta \cap F_\gamma \cap F_\beta)= (F_\alpha \cap F_\beta) \cup (F_\gamma \cap F_\beta)$ (by $F_\alpha \cap F_\beta \subset F_\gamma$)$= (F_\alpha \cup F_\gamma) \cap F_\beta = X \cap F_\beta$ (Since F_α and F_γ are maximal G#rg-closed sets by Theorem 25 (ii), $F_\alpha \cup F_\gamma = X$) $= F_\beta$. That is $F_\beta \cap F_\gamma = F_\beta$ which implies $F_\beta \subset F_\gamma$. Since F_β and F_γ are maximal G#rg-closed sets, we have $F_\beta = F_\gamma$. Therefore $F_\beta = F_\gamma$.

26. Theorem: LetF_α,F_β and F_γ be maximal G#rg-closed sets which are different from each other. Then $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\gamma)$.

Proof: Let $(F_\alpha \cap F_\beta) \subset (F_\alpha \cap F_\gamma)$ which implies $(F_\alpha \cap F_\beta) \cup (F_\gamma \cap F_\beta) \subset (F_\alpha \cap F_\gamma) \cup (F_\gamma \cap F_\beta)$ which implies $(F_\alpha \cup F_\gamma) \cap F_\beta \subset F_\gamma \cap (F_\alpha \cup F_\beta)$. Since by Theorem 25 (ii), $F_\alpha \cup F_\gamma = X$ and $F_\alpha \cup F_\beta = X$ which implies $X \cap F_\beta \subset F_\gamma \cap X$ which implies $F_\beta \subset F_\gamma$. From the definition of maximal G#rg-closed set it follows that $F_\beta = F_\gamma$. This is a contradiction to the fact that F_α, F_β, and F_γ are different from each other. Therefore $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\gamma)$.

27. Theorem: Let F be a maximal G#rg-closed set and x be an element of F. Then F= $\cup$\{E: E is a G#rg-closed set containing x such that F $\cup$ E$\neq$X\}.

Proof: By Theorem 26 and from fact that F is a G#rg-closed set containing x, we have F $\subset$ $\cup$\{E: E is a G#rg-closed set containing x such that F $\cup$E$\neq$ X\}$\subset$ F. Therefore we have the result.

28. Theorem: Let F be a proper nonempty cofiniteG#rg-closed subset. Then there exists (cofinite) maximal G#rg-closed set E such that F $\subset$ E.

Proof: If F is a maximal G#rg-closed set, we may set E=F. If F is not a maximal G#rg-closed set, then there exists (cofinite) G#rg-closed set F_1 such that F$\subset$ $F_1 \neq$X. If F_1 is a maximal G#rg-closed set, we may set E=F_1. If F_1 is not a maximal G#rg-closed set, then there exists a (cofinite) G#rg-closed set F_2such that F$\subset$ $F_1$$\subset$ $F_2$$\neq$X. Continuing this process, we have a sequence of G#rg-closed sets, F $\subset$ $F_1$$\subset$ $F_2$$\subset$ $F_3$$\subset$ F$\subset$... $\subset$ $F_k$$\subset$... .Since F is a cofiniteset, this process repeats only finitely. Then, finally we get a maximal G#rg-closed set E = E_n for some positive integer n.

29. Theorem: Let F be a maximal G#rg-closed set. If x is an element of X−F, then X−F $\subset$ E for any G#rg-closed set E containing x.

Proof: Let F be a maximal G#rg-closed set and x $\in$X−F. E $\subset$ F for any G#rg-closed set E containing x. Then E $\cup$F = X by Theorem 25 (ii). Therefore X−F $\subset$E.

We now introduce minimal G#rg-closed sets and maximal G#rg-open sets in topological spaces as follows,

30. Definition: A proper nonempty G#rg-closed subset F of X is said to be a minimal G#rg-closed set if and only if any G#rg-closed set which is contained in F is $\emptyset$or F.

31. Remark: Every minimal G#rg-closedset need not a minimal closed set as seen from the following example.

32. Example: Let X = \{a, b, c, d, e\} with the topology τ = \{X, Ø, \{a\}, \{d, e\}, \{a, d, e\}\}, then closed sets in X are X, Ø, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}. Minimal closed sets are \{b, c\}. G#rg-closed sets in X are X, ∅, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}. Minimal G#rg-closed sets are \{b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}. Here \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\} are minimal G#rg-closed set but not minimal closed set.

33. Definition: A proper nonempty G#rg-open subset U of a topological space X is said to be a maximal G#rg-open set if and only if any G#rg-open set which contains U is either X or U.

34. Remark: Every maximal G#rg-open set need not maximal open set as seen from the following example.

35. Example: Let X = {a, b, c, d, e} with the topology τ = {X, ∅, {a}, {d, e}, {a, d, e}}, then maximal open sets are {a, d, e} and G#rg-open sets in X are X, Ø, {a}, {b}, {c}, {d}, {e}, {a, d}, {a, e}, {b, d}, {b, e}, {c, d}, {c, e}, {d, e}, {a, d, e}, then maximal G#rg-open sets are {b, d}, {b, e}, {c, d}, {c, e}, {a, d, e}. But{b, d}, {b, e}, {c, d}, {c, e} are maximal G#rg-open sets but not maximal open sets.

36. Theorem: A proper non-empty subset A of X is a maximal G#rg-open set if and only if X−A is a minimal G#rg-closed set.

Proof: Let A be a maximal G#rg-open set. Suppose X−A is not a minimal G#rg-closed set. Then there exists a G#rg-closed set F≠ X−A such that ∅ ≠F⊂ X−A. That is A⊂ X−F and X−F is a G#rg-open set. This contradicts our assumption that A is a minimal G#rg-closed set.

Conversely let X−A be a minimal G#rg-closed set. Suppose A is not a maximal G#rg-open set. Then there exists aG#rg-open set E ≠A such that A⊂E ≠X. That is ∅ ≠X−E ⊂ X−Aand X−E is a G#rg-closed set. This contradicts our assumption that X−A is a minimal G#rg-closed set. Therefore A is a maximal G#rg-closed set.

3. Minimal G#rg-Continuous Maps and Maximal G#rg-Continuous Maps:

1. Definition: Let X and Y are topological spaces. A map f:X → Y is called:

(i) minimal G#rg-continuous (briefly min-G#rg-continuous) if f^{-1}(G) is a G#rg-open set in X for every minimal G#rg-open set G in Y .

(ii) maximal G#rg-continuous (briefly max-G#rg-continuous) if f^{-1}(G) is a G#rg-open set in X for every maximal G#rg-open set G in Y .

2. Theorem: Every G#rg-continuous map is minimal G#rg-continuous but not conversely.

Proof: Let f: X →Y is a G#rg-continuous map. To prove that f is minimal G#rg-continuous. Let G be any minimal G#rg-open set in Y. Since every minimal G#rg-open set is a G#rg-open set, G is a G#rg-open set in Y. Since f is G#rg-continuous, f^{-1}(G) is a G#rg-open set in X. Hence f is a minimal G#rg-continuous map.

3. Example: Consider X = Y = {a, b, c, d} with topologies τ = {X, ∅, {a},{b}, {a, b}, {b, c}, {a, b, c}} and σ = {Y, ∅, {a}, {b, c} {a, b, c}}. Let the function f defined as f(a) = c, f(b) = d, f(c) = a and f(d) =b. Then this function is minimal G#rg-continuous but it is not a G#rg-continuous, since for the open set {b, c} in Y, f^{-1}({b, c}) = {d, a} which is not a G#rg-open set in X.

4. Theorem: Every G#rg-continuous map is maximal G#rg-continuous but not conversely.

Proof:Let f: X →Y is a G#rg-continuous map. To prove that f is maximal G#rg-continuous. Let G be any maximal G#rg-open set in Y. Since every maximal G#rg-open set is a G#rg-open set (by definition ofmaximal G#rg-open set), G is a G#rg-open set in Y. Since f is G#rg-continuous, $f^{-1}(G)$ is a G#rg-open set in X. Hence f is a maximal G#rg-continuous map.

4(i) Example: Consider X = Y = {a, b, c, d} with topologies τ ={X, ∅, {a}, {b}, {a, b}, {a, b, c}} and σ = {Y, ∅, {a}, {b, c}, {a, b, c}}. Let f: X → Y be a function defined as f(a) = a, f(b) = c, f(c) =d and f(d) = a. Then f is a maximal G#rg-continuous map but it is not a G#rg-continuous, since for the open set {b, c} in Y, $f^{-1}(\{b, c\})$ = {d, b} which is not a G#rg-open set in X.

4(ii)Remark: Minimal G#rg-continuous and maximal G#rg-continuous maps are independent of each other.

5. Example: In Example 3, f is a minimal G#rg-continuous but it is not a maximal G#rg-continuous, since {a, c, d} is maximal G#rg-open set in Y but not G#rg-open set in X.

In Example 4(i), f is a maximal G#rg-continuous but it is not a minimal G#rg-continuous, since {d} is minimal open set in Y but not G#rg-open set in X.

6. Remark: MinimalG#rg-continuous and w-continuous(resp. g-continuous) maps are independent of each other.

6(i)Example: (i) Let X = Y = {a, b, c} be with τ = {X, ∅, {a}} and σ = {Y, ∅, {a}, {a, b}}. Define a map f: X → Y by f(a) = c, f(b) = b and f(c) = a. Then f is minimal G#rg-continuous but not w-continuous (resp. g-continuous). Since {a, b} is open set in Y but its inverse is not a w-open (resp. g-open) set in X.

(ii) Let X = Y = {a, b, c} be with τ = {X, ∅, {a, b}} and σ = {Y, ∅, {a}, {a, b}}. Define a map f: X → Y by f(a) = a, f(b) = b and f(c) = a. Then f is w-continuous (resp. g-continuous) but not minimal G#rg-continuous. Since {a} is minimal G#rg-open set in Y but its inverse is not belongs to G#rg-open set in X.

7. Theorem: Let X and Y be the topological spaces. A map f: X → Y is minimal G#rg-continuous if and only if the inverse image of each maximal G#rg-closed set in Y is a G#rg-closed set in X.

Proof: The proof follows from the definition and fact that the complement of minimal G#rg-open set is maximal G#rg-closed set.

8. Theorem: Let X and Y be the topological spaces and A is a nonempty subset of X. If f: X → Y is minimal G#rg-continuous then the restriction map f_A: A → Y is a minimal G#rg-continuous.

Proof: Let f: X → Y is minimal G#rg-continuous map. To prove that f_A: A → Y is a minimal G#rg-continuous. Let N be any minimal G#rg-open set in Y. Since f is minimal G#rg-continuous, $f^{-1}(N)$ is a G#rg-open set in X. But $f_A^{-1}(N) = A \cap f^{-1}(N)$ and $A \cap f^{-1}(N)$ is a G#rg-open set in A. Therefore f_A is a minimal G#rg-continuous.

9. Theorem: If f: X → Y is G#rg-irresolute map and g: Y → Z is minimal G#rg-continuous map, then g∘f: X → Z is a minimal G#rg-continuous.

Proof: Let N be any minimal G#rg-open set in Z. Since g is minimal G#rg-continuous, $g^{-1}(N)$ is a G#rg-open set in Y . Again since f is G#rg-irresolute, $f^{-1}(g^{-1}(N)) = (g\circ f)^{-1}(N)$ is G#rg-open set in X. Hence g∘f is a minimal G#rg-continuous.

10. Theorem: Let X and Y are topological spaces. A map f: X → Y is maximal G#rg-continuous if and only if the inverse image of each minimal G#rg-closed set in Y is a G#rg-closed set in X.

Proof: The proof follows from the definition and fact that the complement of maximal G#rg-open set is minimal G#rg-closed set.

11. Theorem: Let X and Y be the topological spaces and let A be a nonempty subset of X. If f: X → Y is maximal G#rg-continuous then the restriction map f_A: A → Y is a maximal G#rg-continuous.

Proof:Let f: X → Y is maximal G#rg-continuous map. To prove that f_A: A → Y is a maximal G#rg-continuous. Let N be any maximal G#rg-open set in Y. Since f is maximal G#rg-continuous, $f^{-1}(N)$ is a G#rg-open set in X. But $f_A^{-1}(N) = A\cap f^{-1}(N)$ and $A\cap f^{-1}(N)$ is a G#rg-open set in A. Therefore f_A is a maximal G#rg-continuous.

12. Definition:Let X and Y are topological spaces. A map f: X → Y is called:
(i) minimal G#rg-irresolute (briefly min-G#rg-irresolute) if $f^{-1}(G)$ is minimal G#rg-open set in X for every minimal G#rg-open set G in Y .
(ii) maximal G#rg-irresolute (briefly max-G#rg-irresolute) if $f^{-1}(G)$ is maximal G#rg-open set in X for every maximal G#rg-open set G in Y.

13. Theorem: If f: X → Y is G#rg-irresolute map and g: Y → Z is maximal G#rg-continuous map, then g∘f : X → Z is a maximal G#rg-continuous.

Proof:Let N be any maximal G#rg-open set in Z. Since g is maximal G#rg-continuous, $g^{-1}(N)$ is a G#rg-open set in Y . Again since f is G#rg-irresolute, $f^{-1}(g^{-1}(N)) = (g\circ f)^{-1}(N)$ is G#rg-open set in X. Hence g∘f is a maximal G#rg-continuous.

14. Theorem: Every minimal G#rg-irresolute map is minimal G#rg-continuous map but conversely.

Proof: Let f: X → Y be a minimal G#rg-irresolute map. Let N be any minimal G#rg-open set in Y. Since f is minimal G#rg-irresolute, $f^{-1}(N)$ is a minimal G#rg-open set in X. That is $f^{-1}(N)$ is a G#rg-open set in X. Hence f is a minimal G#rg-continuous.

15. Example: Consider X = Y = {a, b, c} with topologies τ ={X, ∅, {a}} and σ = {Y, ∅, {a}, {a, b}. Let f: X → Y be a function defined as f(a) = c, f(b) = c, and f(c) = a. Then f is a minimal G#rg-continuous map but it is not a minimal G#rg-irresolute function, since for the minimal G#rg-open set {c} in Y, $f^{-1}(\{c\}) = \{a, b\}$ which is not a minimal G#rg-open set in X.

16. Theorem: Every maximal G#rg-irresolute map is maximal G#rg-continuous map but not conversely.

Proof: Similar to that of Theorem 15. Let f: X → Y be a maximal G#rg-irresolute map. Let N be any maximal G#rg-open set in Y. Since f is maximal G#rg-irresolute, $f^{-1}(N)$ is a maximal G#rg-open set in X. That is $f^{-1}(N)$ is a G#rg-open set in X. Hence f is a maximal G#rg-continuous.

17. Example: Consider X = Y = {a, b, c, d} with topologies τ ={X, ∅, {a}, {b}, {a, b}, {a, b, c}} and σ = {Y, ∅, {a}, {b, c}, {a, b, c}}. Let f: X → Y be a function defined as f(a) = b, f(b) = d, f(c) =a and f(d) = d. Then f is a maximal G#rg-continuous map but it is not a maximal G#rg-irresolute function, since for the maximal G#rg-open set {a, b, c} in Y, f^{-1}({a, b, c}) = {a, c} which is not a maximal G#rg-open set in X.

18. Remark: Maximal G#rg-irresolute and minimal G#rg-irresolute maps are independent of each other.

19. Example:(i) Consider X = Y = {a, b, c} with topologies τ = {X, ∅, {a}, {b, c}} and σ = {Y, ∅, {a}}. Let f: X → Y be a function defined as f(a) = b, f(b) = c and f(c) = a. Then f is maximal G#rg-irresolute but it is not minimal G#rg-irresolute, since the minimal G#rg-open set {c} in Y, f^{-1}({c}) = {b} which is not a minimal G#rg-open set in X.

(ii) Consider X = Y = {a, b, c} with topologies τ ={X, ∅, {a}} and σ = {Y, ∅, {a}, {b, c}}. Let f: X → Y be an identity function. Then f is a minimal G#rg-irresolute map but it is not a maximal G#rg-irresolute, since for the maximal G#rg-open set {b, c} in Y, f^{-1}({b, c}) = {b, c} which is not G#rg-open set in X.

20. Theorem: Let X and Y be the topological spaces. A map f : X →Y is minimal G#rg-irresolute if and only if the inverse image of each maximal G#rg-closed set in Y is a maximal G#rg-closed set in X.

Proof: The proof follows from the definition and fact that the complement of minimal G#rg-open set is maximal G#rg-closed set.

21. Theorem: If f: X → Y and g: Y → Z are minimal G#rg-irresolutemaps, then g∘f: X → Z is a minimal G#rg-irresolute map.

Proof: Let N be any minimal G#rg-open set in Z. Since g is minimal G#rg-irresolute, $g^{-1}(N)$ is a minimal G#rg-open set in Y . Again since f is minimalG#rg-irresolute, $f^{-1}(g^{-1}(N)) = (g\circ f)^{-1}(N)$ is a minimal G#rg-open set in X. Therefore g∘f is a minimal G#rg-irresolute.

22. Theorem: Let X and Y be topological spaces. A map f: X → Y is maximal G#rg-irresolute if and only if the inverse image of each minimal G#rg-closed set in Y is a minimal G#rg-closed set in X.

Proof: The proof follows from the definition and fact that the complement of maximal G#rg-open set is minimal G#rg-closed set.

23. Theorem: If f: X → Y and g: Y → Z are maximal G#rg-irresolute maps, then g∘f: X → Z is a maximal G#rg-irresolute map.

Proof:Let N be any maximal G#rg-open set in Z. Since g is maximal G#rg-irresolute, $g^{-1}(N)$ is a maximal G#rg-open set in Y . Again since f is maximal G#rg-irresolute, $f^{-1}(g^{-1}(N)) = (g\circ f)^{-1}(N)$ is a maximal G#rg-open set in X. Therefore g∘f is a maximal G#rg-irresolute.

24. Definition:Let X and Y are topological spaces. A map f: X → Y is called:

(i) minimal-maximal G#rg-continuous (briefly min-max G#rg-continuous) if $f^{-1}(G)$ is maximal G#rg-open set in X for every minimal G#rg-open set G in Y.

(ii) maximal-minimal G#rg-continuous (briefly max-min G#rg-continuous) if $f^{-1}(G)$ is minimal G#rg-open set in X for every maximal G#rg-open set G in Y.

25. Theorem: Every minimal-maximal G#rg-continuous map is minimal G#rg-continuous map but not conversely.

Proof: Let f: X → Y be a minimal-maximal G#rg-continuous map. Let N be any minimal G#rg-open set in Y. Since f is minimal-maximal G#rg-continuous, $f^{-1}(N)$ is a maximal G#rg-open set in X. Since every maximal G#rg-open set is a G#rg-open set, $f^{-1}(N)$ is a G#rg-open set in X. Hence f is a minimal G#rg- continuous.

26. Example: Consider X = Y = {a, b, c} with topologies τ = {∅, X, {a}, {b, c}} and μ = {∅, Y, {a}, {b}, {a, b}}. Let f: X → Y be a map defined as f(a) = b, f(b) = c and f(c) = a. Then f is minimal G#rg-continuous but it is not a minimal-maximal G#rg-continuous, since for the minimal G#rg-open sets {a}, {c} in Y, f^{-1}({a})= {c}, f^{-1}({c}) = {b}, which are not a maximal G#rg-open set in X.

27. Theorem: Every maximal-minimal G#rg-continuous map is maximal G#rg-continuous map but not conversely.

Proof:Let f: X → Y be a maximal-minimal G#rg-continuous map. Let N be any maximal G#rg-open set in Y. Since f is maximal-minimal G#rg-continuous, $f^{-1}(N)$ is a minimal G#rg-open set in X. Since every minimal G#rg-open set is a G#rg-open set, $f^{-1}(N)$ is a G#rg-open set in X. Hence f is a maximal G#rg- continuous.

28. Example:Consider X = Y = {a, b, c} with topologies τ = {∅, X, {a}, {b}, {a, b}} and μ = {∅, Y, {a}, {a, c}}. Let f: X → Y be a map defined as f(a) = a, f(b) = c and f(c) = b. Then f is maximal G#rg-continuous but it is not a maximal-minimalG#rg-continuous, since for the maximal G#rg-open set {a, c} in Y, f^{-1}({a, c}) = {a, b}, which is not a minimal G#rg-open set in X.

References:

[1] Nakaoka F. and Oda N., Some applications of minimal open sets, *Int. J. Math. Math. Sci., 27, 8 (2001), 471-476.*

[2] Nakaoka F. and Oda N., Some properties of maximal open sets, *Int. J. Math. Sci., 21 (2003), 1331-1340.*

[3] Benchalli S. S., Basavaraj M. Ittanagi and R. S. Wali, On minimal open sets and maps in topological spaces, *J. Comp. and Math. Sci., 2 2 (2011), 208-220.*

[4] Nakaoka F. and Oda N., On minimal closed sets, *Proceedings of Topological Spaces Theory and its Applications, 5 (2003), 19-21.*

[5] Jenifer J. Karnel and Holabasayya I. Sankannavar, Some studies on generalized #rg-closed sets, *J.Env.Sci.Comp.Sci.Eng.Tech, 10(2)2021, 352-361.*

[6] Jenifer J. Karnel and Holabasayya I. Sankannavar, Some studies on generalized #rg-open sets and generalized #rg-operators, *Mathematical Sciences International Research Journal, 10(2)2021, 9-18.*