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Section A-Research paper

A Fractional Differential Approach to Plant-Pest Dynamic Model with Infected Pest

Anil Kumar Shukla[1,3], Joydip Dhar[2] , Rajshree Mishra[3]

[1]Govt. P. G. College Guna, Madhav Rao Sciendia Road, 473001, Guna, India

[2]ABV – IIITM, Banmor Road, Gwalior, 474015, India

[3]SMS Govt Model Science College Gwalior, 474001

Email addresses: anilshukla763@gmail.com (Anil Kumar Shukla), jdhar.iiitmg@gmail.com (Joydip Dhar), rajshreemishraa@gmail. com (Rajshree Mishra)

Abstract

In the modern era, plant pest is a significant problem. Many options are available, including chemical remedies and ones based on natural enemies. In this work, we discuss the dynamics of infected pests on plants. We have suggested that Belington-DeAngeli functional responses in pests cause infection spread. Because natural enemies are disappearing at an alarming rate and chemicals are hazardous to both plant life and humans, that's the reason for choosing the model with diseased pests. In this study, the existence, uniqueness, boundedness, and non-negativity for the solution of the model have been carried out. Analysis of the equilibrium point's local and global stability and numerical simulation have been done at the end of this work.

Keywords: Fractional order Derivative, Three species dynamics, Healthy Pest, Diseased Pest

1. Introduction

Recently an important region, "Pest with some disease," captivated investigators because the disease leads to the depletion of the pest population. Researchers were initially concerned about the plant pest issue because of its extreme importance in biological sciences. The most widely used method to address this critical issue is chemical intervention, but due to the harmful effects that chemicals have on both plant and human species, careful research is being done to find better alternatives to chemicals Recently an important region "Pest with some disease" captivated investigators because it leads to the depletion of the pest population. in this context, Fred Gould wrote a paper [1]in the year 2008, on genetic paste management in which he introduces how genetical changes control the paste population which is not harmful to the human species, 2017 Tim Harvey et al. wrote a paper [2], on genetic pest management by DNA sequencing and stated that this technique is very efficient for the pest management and in 2023 AlperenKutalm et al. investigated in his paper [3], that the entomopathogenic fungi are a biological control disease for the paste Tropinota, Hirta, and the apple scab. In 2014 a mathematical model on genetic paste management is reported by Alphey et al. [4], a notable work has been reported by Kumar et al. in 2017 [5] and in 2020 Vella et al. wrote a paper [6] on genetic paste management by female-specific approach. As of now, ODE is the foremost technique used to prepare various plant pest models using numerous kinds of parameters like delay or without delay harvesting, etc., [7-9]. The fractional differentiation approach is a cutting-edge field of research because it produces unique results in the system known as the memory effect, which is a very advantageous standard for resolving plant pest issues.

Weihua Deng explained the memory effect that fractional differential equations exhibit in his research study [10]. In 2017 Comlekoglu T et al. write the memory effect of fractional differential equation in cardiomyocyte [11] Consequently, tremendous work has been reported in this sector because of the exceptional attainment of these tactics [12-18]. To address the inadequacies in models created for plant-pest relationships, several scientists have experimented repeatedly corrected models by examining divisor parts of a real-world problem. The proposed model is significantly more capable and efficient than earlier models. In this study effort, we discussed a fractional-order differential model for plant immature-mature pests and natural enemies.

1.1. Modelling Methodology

In this work, we investigated the dynamics of the food chain of plant pests with infection in prey. We presented the following mathematical strategy in this instance:

1 Plant ($p(t)$), Healthy Pest $\left(h_p(t)\right)$, and Infected pest$\left(i_p(t)\right)$ are the three categories of species.

2 The plants develop logistically with an inherent advancement rate of r and a carrying capacity of k. When there is no healthy pest population, the plants' per capita advance rate is $r * p\left(1-\frac{p}{k}\right)$.

3 α is the rate at which healthy pest harvest plants with functional type-I response.

4 The Bellington-DeAngelli functional response states that the healthy pest becomes infected at the rate of β, where θ is the conversion rate that ranges from 0 to 1 by which healthy pests are increasing while cropping plants. d is the healthy pest death rate.

5 the natural death rate of infected pest is d_1.

Plant Healthy and infected pest model-

$$
\begin{aligned}
{}^{c}D^{\delta} p &= rp\left(1-\frac{p}{k}\right)-\alpha p h_p \\
{}^{c}D^{\delta} h_p &= \theta\alpha p h_p-\frac{\beta h_p i_p}{i_p+c_1+b_1 h_p}-d h_p \\
{}^{c}D^{\delta} i_p &= \frac{\beta h_p i_p}{i_p+c_1+b_1 h_p}-d_1 i_p
\end{aligned}
$$

With initial conditions:

$p(0)>0, h_p(0)>0$, and $i_p(0)>0$., The following summarises our research: We discussed a few basic definitions of fractional derivatives in the introduction. The results for the existence and uniqueness, non-negativity, and boundedness of the system (1.1) are generated in the major findings section. For the system (1.1), the equillibrium points and their stability analysis are also performed. Using the fractional differential equations Matlab code fde12, numerical analysis is conducted at the end of this study.

2. Preliminaries

This part will go through some basic fractional calculus results and terminology that will be used throughout the investigation.

Definition 2.1. The fractional integral of a function ϵ with order $\delta>0$ lower bound zero is defined as follows [19]:

$$
I^{\delta}\epsilon(x)=\frac{1}{\Gamma(\delta)}\int_0^x (x-\eta)^{\delta-1}\epsilon(\eta)d\eta, \quad x>0
$$

where $I^0\epsilon(x):=\epsilon(x)$ and the Euler Gamma function is denoted by $\Gamma(\cdot)$. For $b>0$, this fractional integral satisfies the conditions $I^{\delta}\circ I^{b}=I^{\delta+b}$.

Definition 2.2. The Riemann-Liouville fractional derivative of a function ϵ with the lower limit zero of order $\delta>0$ is given by [20]

$$D^{\delta}\epsilon(x)=\frac{1}{\Gamma(n-\delta)}\frac{d^n}{dx^n}\int_0^x (x-\eta)^{n-\delta-1}\epsilon(\eta)d\eta,$$

where $n-1<\delta<n$ and $n\in\mathbb{N}$, up to order $(n-1)$, the function $\epsilon(t)$ has an absolutely continuous derivative. Moreover, $D^0\epsilon(x)=\epsilon(x)$ and $D^{\delta}I^{\delta}\epsilon(x)=\epsilon(x)$ for $t>0$.

Definition 2.3. The Caputo fractional derivative of a function $\epsilon\in C^n([0,\infty))$ with the lower limit zero of order $\delta>0$ is given by [20]

$${}^{c}D^{\delta}\epsilon(x)=\frac{1}{\Gamma(n-\delta)}\int_0^x (x-\eta)^{n-\delta-1}\frac{d^n}{d\eta^n}\epsilon(\eta)d\eta, \text{ where } n-1<\delta<n \text{and } n\in\mathbb{N}.$$

Lemma 2.1. [21] Consider the system:

$${}^{c}D^{\delta}f(t)=g(f,t)$$

with the initial condition: $f(t_0)=f_{t_0}$, where $\delta\in(0,1], g:\Gamma\times[t_0,\infty)\to\mathbb{R}^n, \Gamma\subset\mathbb{R}^n$, if $g(f,t)$ satisfies the local Lipschitz condition with respect to f, then there exists a unique solution of (2.1) on $\Gamma\times[t_0,\infty)$.

3.Main Result

3.1 Theorem of Existence and Uniqueness

Theorem 3.1. The sufficient condition for the existence and uniqueness of the solution of the system (1.1) in the region $\Gamma\times[t_0,T)$ Here $\Gamma=\{(p,h_p,i_p)\in\mathbb{R}^3:\max\{|p|,|h_p|,|i_p|\}<L\}, T<\infty, \psi=(p,h_p,i_p), \bar{\psi}=(\bar{p},\bar{h}_p,\bar{i}_p)$ with the initial condition $\psi(t_0)=\left(p_{t_0},h_{p_{t_0}},i_{p_{t_0}}\right)\in\Gamma$, is $H<1$and $H=\frac{T^{\delta}}{\Gamma\delta+1}\max\{H_1,H_2,H_3\}$, where

$$H_1=r+\frac{2rL}{k}+(\theta+1)\alpha L$$
$$H_2=(\theta+1)\alpha L+d+\frac{2\beta L}{c_1},$$
$$H_3=\frac{2\beta L}{c_1}+d_1.$$

Proof. The contraction mapping principle, which has been utilized in numerous studies, forms the foundation for this theorem's proof [22,23], now one can rewrite the fractional order system (1.1) as

$$D^{\delta}\psi(t)=F(\psi(t)), t\in(0,T],$$
$$\psi(0)=\psi_0.$$

$$\text{Where,}\psi=\begin{pmatrix}p\\h_p\\i_p\end{pmatrix},\psi_0=\begin{pmatrix}p_0\\h_{p_0}\\i_{p_0}\end{pmatrix},F(\psi)=\begin{pmatrix}rp\left(1-\frac{p}{k}\right)-\alpha ph_p\\ \theta\alpha h_p-\frac{\beta h_p i_p}{i_p+c_1+b_1h_p}-dh_p\\ \frac{\beta h_p i_p}{i_p+c_1+b_1h_p}\end{pmatrix}$$

The solution of the system of equation (1.1) is, $\psi(t)=\psi_0+\frac{1}{\Gamma(\delta)}\int_0^t (t-x)^{\delta}F(\psi(x))dr=R(\psi)$

Thus,$R(\psi)-R(\bar{\psi})=\frac{1}{\Gamma(\delta)}\int_0^t (t-x)^{\delta}(F(\psi(x))-F(\bar{\psi}(x)))dx$

Then from the property of modulus of integration one gets

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$$|R(\psi)-R(\bar{\psi})| \leq \frac{1}{\Gamma(\delta)}\int_0^t (t-x)^{\delta} \mid (F(\psi(x))-F(\bar{\psi}(x)) \mid dx$$

This gives the result, $|R(\psi)-R(\bar{\psi})| \leq \frac{T^{\delta}}{\Gamma\delta+1} max\{H_1,H_2,H_3\}|\psi-\bar{\psi}|$

It follows that the Lipschitz condition is satisfied by the function $R(\psi)$. If $H<1$, then the mapping $\psi = R(\psi)$ is a contraction mapping, proving that there is only one solution to the system of equations (1.1).

3.2. The solution is positive and uniformly bounded

Theorem 3.2. The Solutions of the system (1.1)which starts in $\mathbb{R}_+^3$ are affirmative and uniform bounded.

Proof. Suppose $W(t) = \theta p(t) + h_p(t) + i_p(t)$

then

$$\begin{aligned}
{}^{c}D^{\delta}W(t) &= \theta\, {}^{c}D^{\delta}p(t) + {}^{c}D^{\delta}h_p(t) + {}^{c}D^{\delta}i_p(t)\\
&= \theta rr\left(1-\frac{p}{k}\right) - \theta\alpha p h_p + \theta\alpha p h_p - \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - d h_p + \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - d_1 i_p,\\
{}^{c}D^{\delta}W(t) + mW(t) &= -\frac{\theta r}{p}(p-k)^2 + \theta(m-r)p + (m-d)h_p + (m-d_1)i_p + \theta rk,
\end{aligned}$$

one can use the results of the theorem-(2) of the paper [22]. Let us assume $m < min\{r, d, d_1\}$, then

$${}^{c}D^{\delta}W(t) + mW(t) \leq \theta rk$$

Now, using the fractional order comparison theorem as discussed in [22, 24], one reaches

$$0 \leq W(t) \leq W(0)E_{\delta}\left(-mt^{\delta}\right) + \left(\theta rk\left(t^{\delta}\right)E_{\delta,\delta+1}\left(-m(t)^{\delta}\right)\right.$$

here $E_{\delta,\delta+1}$ is a mittag leffler function so on taking $t \to \infty$ one gets, $0 \leq W(t) \leq (\theta rk)$

hence, the solutions of system of fractional differential equation (1.1) begins in $\mathbb{R}_+^3$ are uniformly bounded with in the region W_1 defined as

$$W_1 = \left\{\left(p, h_p, i_p\right) \in \mathbb{R}_+^3 : W(t) \leq (\theta rk + \delta), \delta > 0\right\}$$

We shall now demonstrate that the fractional order system (1.1) solution is not negative (Positive).

Consider the first equation of system of equation (1.1).

$${}^{c}D^{\delta}p = rp\left(1-\frac{p}{k}\right) - \alpha p h_p$$

from equation (4.1) and (4.2) and considering $\delta \to 0$ one gets

$$W(t) = \theta p(t) + h_p(t) + i_p(t) \leq (\theta rk) = l_1$$

from (4.3) and (4.4) one gets

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$$
\begin{aligned}
{}^{c}D^{\delta}p &\geq pr\left(1-\frac{l_1}{k}\right)-\alpha l_1 p,\\
{}^{c}D^{\omega}x &\geq \alpha p,
\end{aligned}
$$

Where, $\alpha_1 = \left(r\left(1-\frac{l_1}{k}\right)-\alpha l_1\right)$

from comparison theorem of [22,24] one gets, $p \geq p_0 E_{\delta,1}\left(\alpha_1 t^{\delta}\right)$, hence $p \geq 0$.

Now on taking equation-(2) of system (1.1) and from (4.1) one gets

$$
\begin{aligned}
{}^{c}D^{\delta}h_p &= \theta\alpha p h_p - \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - dh_p\\
{}^{c}D^{\delta}h_p &\geq -\frac{\beta l_1}{c_1}h_p - dh_p\\
{}^{c}D^{\delta}h_p &\geq \left(-\frac{\beta l_1}{c_1}-d\right)h_p, \geq -l_2 h_p
\end{aligned}
$$

where $l_2 = \left(\frac{\beta l_1}{c_1}+d\right)$, Hence

$$
\begin{aligned}
h_p &\geq h_{p_0}E_{\delta,1}\left(-l_2 t^{\delta}\right)\\
&\Rightarrow h_p \geq 0
\end{aligned}
$$

Now considering the equation-(3) of system of equation (1.1) one gets

$$
\begin{aligned}
{}^{c}D^{\delta}i_p &= \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - d_1 i_p,\\
{}^{c}D^{\delta}i_p &\geq \left(\frac{\beta 0 i_p}{i_p + c_1 + b_1 h_p}\right) - d_1 i_p,\\
&\geq -d_1 i_p,\\
&\Rightarrow i_p \geq i_{p0}E_{\delta,1}\left(-d_1 t^{\delta}\right),\\
&\Rightarrow i_p \geq 0.
\end{aligned}
$$

Hence the system of equation (1.1) has positive definite solutions.

4. Analysis of the Stationary Point's Stability

4.1 Stationary points

1 The stationary point $E_0(0,0,0)$ always present.

2 The axial stationary point $E_1(k,0,0)$ is present. 3. The infection free stationary Point $E_2\left(p', h_p\ ', 0\right)$ is exist only when ($\mathbb{R}_0 > 1$).Where,

$$
\begin{aligned}
p' &= \frac{d}{\theta\alpha},\\
h_p' &= \frac{r}{\alpha}\left(1-\frac{d}{\theta\alpha k}\right).
\end{aligned}
$$

4 The coexisting stationary point $E_3(p^*, h_p{}^*, i_p{}^*)$ exist only when $\beta + d > d_1(b_1 + c_1), h_p{}^* > 0$, and $i_p{}^* > 0$.Where,

$$p^* = \frac{\left[\frac{(\beta+d-d_1b_1)}{r\theta\alpha} + k\right] \pm \sqrt{\left[\frac{(-\beta+d_1b_1-d)}{r\theta\alpha} + k\right]^2 + \frac{4d_1c_1k}{r\theta}}}{2},$$

$$h_p{}^* = \frac{-d_1c_1}{\theta\alpha p - \beta + d_1b_1 - d},$$

$$i_p{}^* = \frac{(\beta - d_1b_1)h_p{}^* - d_1c_1}{d_1}.$$

4.2. Basic Reproduction Number

For finding equilibrium points and their stability we will introduce basic reproduction number with the help of pest free Stationary point of system of equation (1.1).

Theorem 4.1. The basic reproduction number $\mathbb{R}_0$ for the system of equation (1.1) is given by $\mathbb{R}_0 = \frac{\theta\alpha k}{d}$.

Proof. Rewriting the given system of equation (1.1)

$$^cD^\delta h_p = \theta\alpha p h_p - \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - d h_p$$

$$^cD^\delta i_p = \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - d_1 i_p$$

$$^cD^\delta p = rp\left(1 - \frac{p}{k}\right) - \alpha p h_p.$$

system (5.1) can be written as, $^cD^\delta P(t) = D(X) - F(X)$, Where,

$$D(X) = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} \theta\alpha p h_p \\ 0 \\ 0 \\ 0 \end{pmatrix}, F(X) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} + d h_p \\ -\frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} + d_1 i_p \\ -rp\left(1 - \frac{p}{k}\right) + \alpha p h_p \end{pmatrix}$$

Now the Matrices $M(X)$ and $N(X)$ can be defined as$M(X) = \begin{pmatrix} \frac{\partial D_1}{\partial h_p} & \frac{\partial D_1}{\partial i_p} & \frac{\partial D_1}{\partial p} \\ \frac{\partial D_2}{\partial h_p} & \frac{\partial D_2}{\partial i_p} & \frac{\partial D_2}{\partial p} \\ \frac{\partial D_3}{\partial h_p} & \frac{\partial D_3}{\partial i_p} & \frac{\partial D_3}{\partial p} \end{pmatrix}, N(X) = \begin{pmatrix} \frac{\partial F_1}{\partial h_p} & \frac{\partial F_1}{\partial i_p} & \frac{\partial F_1}{\partial p} \\ \frac{\partial F_2}{\partial h_p} & \frac{\partial F_2}{\partial i_p} & \frac{\partial F_2}{\partial p} \\ \frac{\partial F_3}{\partial h_p} & \frac{\partial F_3}{\partial i_p} & \frac{\partial F_3}{\partial p} \end{pmatrix}$

$$M(X) = \begin{pmatrix} \theta\alpha p & 0 & \theta\alpha h_p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N(X) = \begin{pmatrix} \frac{\beta i_p(i_p + c_1)}{(i_p + c_1 + c_1 h_p)^2} - d & \frac{\beta h_p(b_1 h_p + c_1)}{(i_p + c_1 + b_1 h_p)^2} & 0 \\ -\frac{\beta i_p(i_p + c_1)}{(i_p + c_1 + b_1 h_p)^2} & -\frac{\beta h_p(b_1 h_p + c_1)}{(i_p + c_1 + b_1 h_p)^2} + d_1 & 0 \\ \alpha p & 0 & -rp + \frac{2rp}{k} + \alpha h_p \end{pmatrix}.$$

To determine the characteristic values of the stationary point for pest extinction $E_1(k,0,0)$, the equation $|MN^{-1}-\omega I|=0$ has to be solved, ω is the characteristic value I is the identity matrix, MN^{-1} is the next generation matrix for model (5.1), ω_1,ω_2, and ω_3 can be computed as $\omega_1=0,\omega_2=0$, and $\omega_3=\frac{\alpha\theta k}{d}$. The matrix's MN^{-1} spectral radius is $\rho(M.N^{-1})=\max(\omega_i), i=1,2,3$. Using theorem-3 in [22] model's (1.1) fundamental replication number or basic reproduction number is $\mathbb{R}_0=\frac{\alpha\theta k}{d}$.

4.3. Local Stability

One can see from the paper [25] that the local stability can be obtained by characteristics values of the following Jacobian matrix for the given system of equation (1.1)

$$J(p,h_p,i_p)=\begin{pmatrix} r\left(1-\frac{2p}{k}\right)-\alpha h_p & -\alpha p & 0 \\ \theta\alpha h_p & \theta\alpha p-\frac{\beta i_p(c_1+i_p)}{(i_p+c_1+b_1h_p)^2}-d & -\frac{\beta h_p(b_1h_p+c_1)}{(i_p+c_1+b_1h_p)^2} \\ 0 & \frac{\beta i_p(c_1+i_p)}{(i_p+c_1+b_1h_p)^2} & \frac{\beta h_p(b_1h_p+c_1)}{(i_p+c_1+b_1h_p)^2}-d_1 \end{pmatrix}.$$

Theorem 4.2. The stationary point $E_0(0,0,0,0)$ of system (1.1) is unstable.

Proof. The Jacobian matrix for equilibrium point E_0 is

$$J(0,0,0)=\begin{pmatrix} r & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -d_1 \end{pmatrix}$$

The characteristic values of the system are $\omega_1=r,\omega_2=-d$, and $\omega_3=-d_1$ There fore $|\arg(\omega_1)|=0<\frac{\delta\pi}{2}$ where $0<\delta<1$ hence E_0 is unstable.

Theorem 4.3. The axial stationary point $E_1(k,0,0)$ is locally asymptotically stable, when $\mathbb{R}_0<1$ holds.

Proof. The Jacobian matrix for E_1 is, $J(k,0,0)=\begin{pmatrix} -r & -\alpha k & 0 \\ 0 & \theta\alpha k-d & 0 \\ 0 & 0 & -d_1 \end{pmatrix}$

. The characteristic values for the stationary point E_1 are $\omega_1=-r,\omega_2=\theta\alpha k-d$, and $\omega_3=-d_1$. The argument of the characteristic values is$|\arg(\omega_1)|=\pi>\frac{\delta\pi}{2},|\arg(\omega_2)|=\pi>\frac{\delta\pi}{2}$, only when $\mathbb{R}_0<1$ hold and $|\arg(\omega_3)|=\pi>\frac{\delta\pi}{2}$. Hence the stationary point E_1 is locally asymptotically stable when $\mathbb{R}_0<1$ holds.

Theorem 4.4. The infection free equilibrium points$E_2(p',h_p',0)$ if exist then it is locally stable when the set of conditions $\mathbb{R}_0<\frac{\frac{r\beta}{\alpha}-b_1d_1}{\frac{r\beta}{\alpha}-d_1b_1-d_1c_1}$ holds.

Proof. The Jacobian matrix for the equilibrium point E_3 is

$$J(p',h_p',0)=\begin{pmatrix} -r\frac{d}{\theta\alpha k} & \frac{-d}{\theta} & 0 \\ r\left(1-\frac{d}{\theta\alpha k}\right) & 0 & \frac{\frac{-r\beta}{\alpha}\left(1-\frac{d}{\theta\alpha k}\right)}{c_1+b_1\left(1-\frac{d}{\theta\alpha k}\right)} \\ 0 & 0 & \frac{\frac{-}{\alpha}\left(1-\frac{d}{\theta\alpha k}\right)}{c_1+b_1\left(1-\frac{d}{\theta\alpha k}\right)}-d_1 \end{pmatrix}$$

The characteristic values for the given Jacobian matrix are $\omega_1 = \frac{\left(\frac{r\beta}{\alpha}-d_1b_1\right)\left(1-\frac{1}{\mathbb{R}_0}\right)-d_1c_1}{c_1+b_1\left(1-\frac{1}{\mathbb{R}_0}\right)}$, and the other eigenvalues of the matrix are given by the following quadratic equation

$$\omega^2 + A_1\omega + A_2$$

Where,

$$A_1 = r\frac{d}{\theta\alpha k}$$
$$A_2 = r\left(1-\frac{d}{\theta\alpha k}\right),$$

according to the Routh-Hurwitz criteria for the fractional differential equation as discussed in [25] if $A_1 > 0$, and $A_1A_2 > 0$ then the characteristic values of (5.2) have negative real part. Hence the argument of obtained eigenvalues $|\arg(\omega_1)| = \pi > \frac{\delta\pi}{2}$ when $\mathbb{R}_0 < \frac{\frac{r\beta}{\alpha}-b_1d_1}{\frac{r\beta}{\alpha}-d_1b_1-d_1c_1}$, and $|\arg(\omega_2)| = \pi > \frac{\delta\pi}{2}, |\arg(\omega_3)| = \pi > \frac{\delta\pi}{2}$. The stationary point E_2 will be locally asymptotically stable when the condition $\mathbb{R}_0 < \frac{\frac{r\beta}{\alpha}-b_1d_1}{\frac{\gamma\beta}{\alpha}-d_1b_1-d_1c_1}$ holds.

Theorem 4.5. The Endemic equilibrium point $E_3(p^*, h_p^*, i_p^*)$ is locally asymptotically stable when following condition $B_0 > 0, B_2 > 0, B_0B_1 > B_2$ holds.

Proof. The Jacobian matrix for the stationary point E_3 is

$$J(p^*, h_p{}^*, i_p{}^*) = \begin{pmatrix} r\left(1-\frac{2p^*}{k}\right)-\alpha h_p{}^* & -\alpha p^* & 0 \\ \theta\alpha h_p{}^* & \theta\alpha p^* - \frac{\beta i_p{}^*(c_1+i_p{}^*)}{(i_p{}^*+c_1+b_1h_p{}^*)^2} - d & -\frac{\beta h_p{}^*(b_1h_p{}^*+c_1)}{(i_p+c_1+b_1h_p)^2} \\ 0 & \frac{\beta i_p{}^*(c_1+i_p{}^{**}}{(i_p{}^*+c_1+b_1h_p{}^*)^2} & \frac{\beta h_p{}^*(b_1h_p{}^*+c_1)}{(i_p{}^*+c_1+b_1h_p)^2} - d_1 \end{pmatrix}$$

the characteristic values for the given Jacobian matrix are given by the following cubic equation.

$$\omega^3 + B_0\omega^2 + B_1\omega + B_2$$
$$D(P) = 18B_0B_1B_2 + (B_0B_1)^2 - 4B_2B_0^3 - 4B_1^3 - 27B_2^2,$$

according to the Routh-Hurwitz criteria for the fractional differential equation as discussed in [25] if $D(P) > 0$ and $B_0 > 0, B_2 > 0, B_0B_1 > B_2$ then the characteristic values of (5.2) have negative real part. Hence the argument of obtained eigenvalues $|\arg(\omega_1)| > \frac{\delta\pi}{2} |\arg(\omega_2)| > \frac{\delta\pi}{2}$, and $|\arg(\omega_3)| > \frac{\delta\pi}{2}$. The stationary point E_3 will be locally asymptotically stable. as in the paper [25] it is given that the Routh-Hurwitz condition is the sufficient condition for the stability of stationary point and the necessary condition for the stability of characteristic values is $B_2 > 0$, and the Routh-Hurwitz condition is$B_0 > 0, B_0B_1 > B_2$,holds.

4.3. Global Stability

Theorem 4.6. The pest extinction stationary point $E_1(k, 0,0)$ is globally asymptotically stable if $\mathbb{R}_0 < 1$.

Proof. Now consider the Lyapunov function as follows:

$$U(p,h_p,i_p) = \theta\left(p - k - k\ln\left(\frac{x}{k}\right)\right) + h_p + i_p$$

by virtue of lemma (3.1) of [?] the ω order derivative of $U(p,h_p,i_p)$ is

$$^{c}D^{\delta}U(p,h_p,i_p) \le \theta\left(1-\frac{k}{p}\right)^{c} D^{\delta}p + \ {}^{c}D^{\delta}h_p + \ {}^{c}D^{\delta}i_p$$

$$\le \theta\left(1-\frac{k}{p}\right)\left\{rp\left(1-\frac{p}{k}\right) - \alpha p h_p\right\} + \left\{\theta\alpha p h_p - \frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - dh_p\right\} + \left\{\frac{\beta h_p i_p}{i_p + c_1 + b_1 h_p} - d_1 i_p\right\}$$

$$\le \frac{-\theta r}{k}(p-k)^2 + (\theta\alpha k - d)h_p - d_1 i_p.$$

When $\mathbb{R}_0 = \frac{r\theta\alpha}{k} < 1$ then $\ {}^{c}D^{\delta}U \le 0$

This shows from lemma (4.6) in [26] the pest extinction stationary point is globally asymptotically stable.

5. Numerical Analysis

In this section we have developed some graphs of the system (1.1) by using Matlab code FDE12 [12].

Table 1: One can use the following values of the parameter as mentioned in the table for generating the Graphs.

Para. ↓	Col1	Col2	Col3
r	0.9	0.9	0.9
k	5	5	5
α	0.05	0.05	0.05
θ	0.02	0.3	0.3
β	0.05	0.05	0.05
c_1	0.3	0.3	0.3
b_1	0.03	0.03	0.03
d	0.05	0.05	0.05
d_1	0.3	0.76	0.3

By using the values of column-1 of Table- 1 one will get the value of $\mathbb{R}_0 = 0.1 < 1$ then we found the graph of pest extinction equilibrium point.

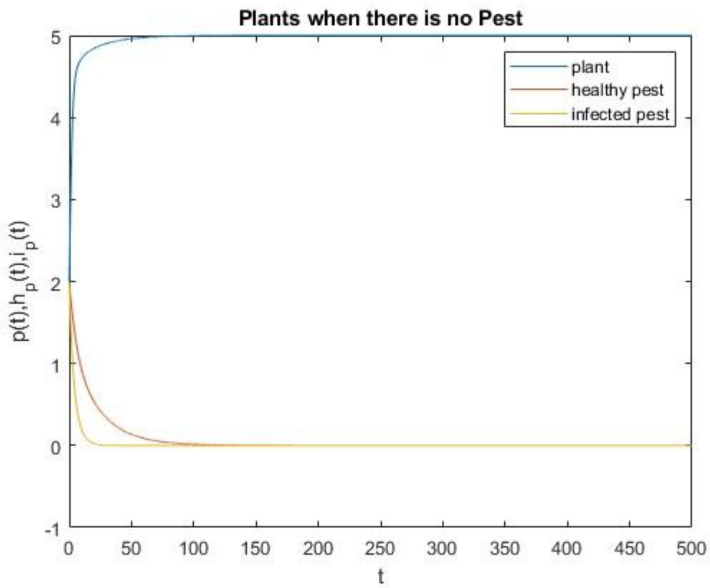


Figure-1(a) $E_1(5,0,0)$ Time series graph of pest extinction stationary point for the order $\delta = 1$.

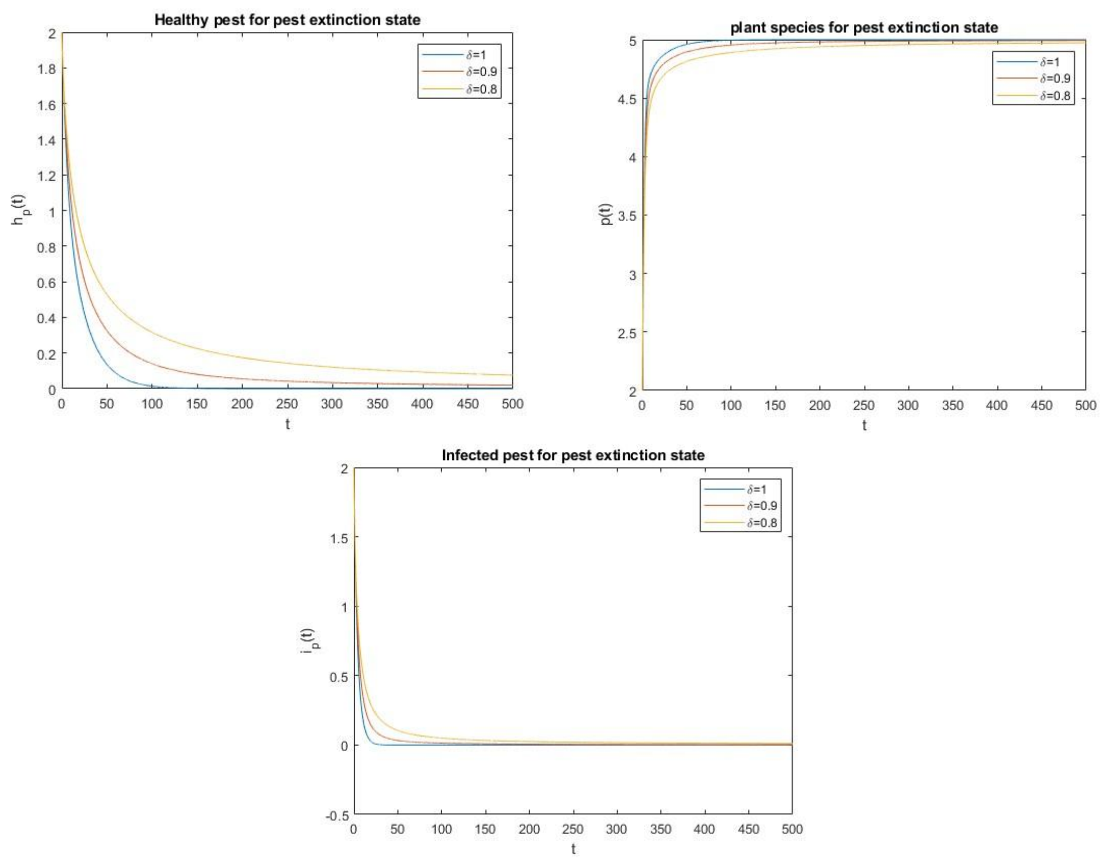


Figure-1(b)-State trajectories graph of system (1.1) for the pest extinction stationary point corresponding to different values of $\delta = 1,0.9,0.80$

One can see from the column-2 of Table-1 that if we consider the value of $\theta = 0.3$ and $d_1 = 0.76$ one can get $\mathbb{R}_0 = 1.5 > 1\$$ then one got the infection free stationary state it can be seen from the following graphs.

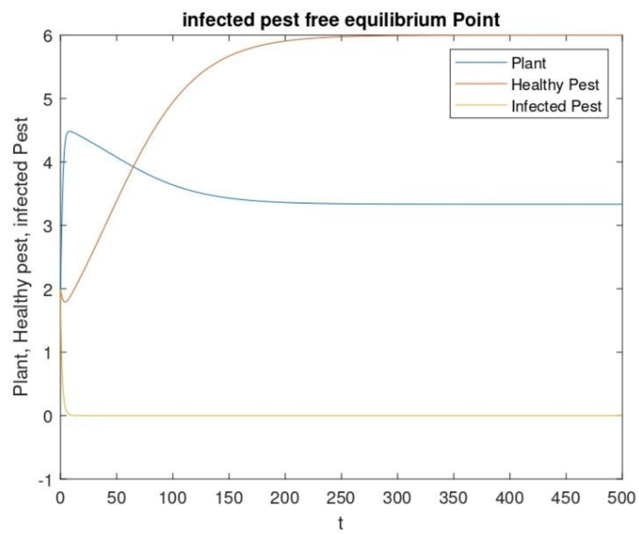
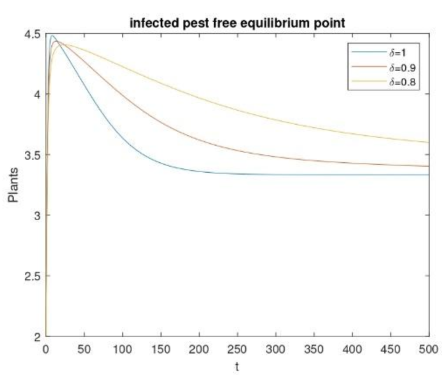
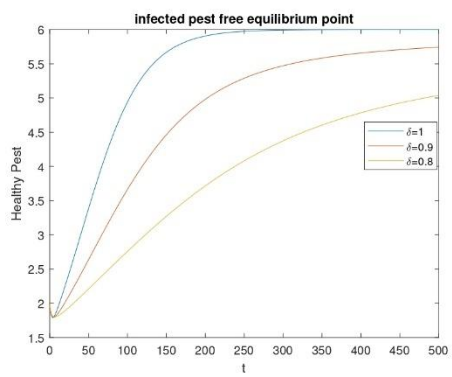


Figure-2(a) Time series graph of infection free stationary point $E_2(3.333,6.00,0)$ for the fractional order $\delta = 1$.





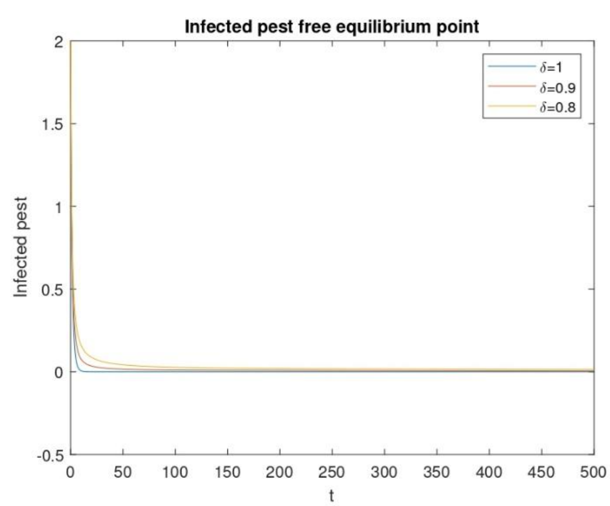


Figure 2(b) variation of state trajectories graph of system (1.1) for natural enemy free equilibrium point corresponding to different values of $\delta = 1, 0.9, 0.8$.

One can observe from the column-3 of Table-1 if we consider the values of $d_1 = 0.3$ and remains the values of parameter unchanged as in column-2 then one gets $\mathbb{R}_0 = 1.5$ and reached to the coexisting stationary state.

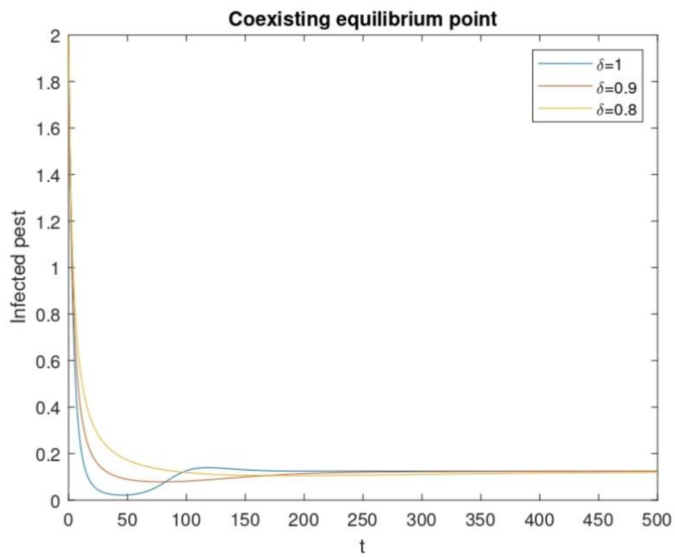


Figure-3(a) Time series graph of coexisting stationary point $E_3(4.1365, 3.109, 0.1248)$.

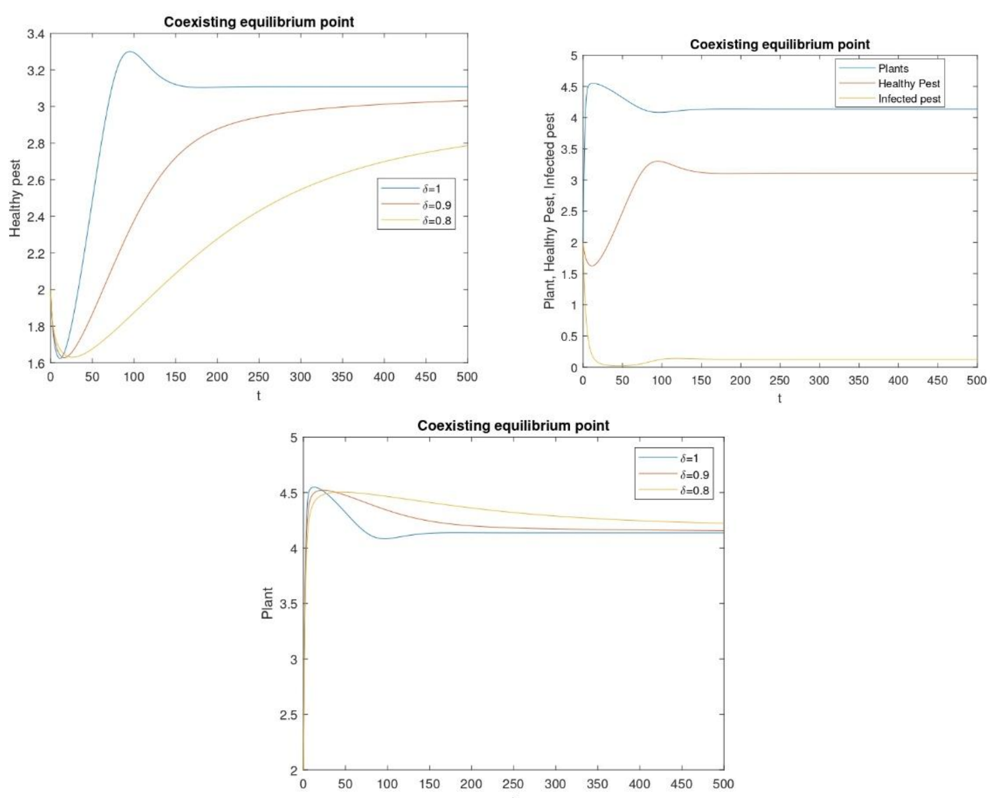


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Figure-3(b) variation of state trajectories graph of system (1.1) for coexisting equilibrium point corresponding to different values of $\delta = 1{,}0.9{,}0.8$.

6. Conclusion

The existence, uniqueness, positivity, and uniform boundedness of the solution to the system of equations governed by (1.1) have all been covered in the Theorems(3.1) and (3.2), respectively, of this study. Theorems (4.2), (4.3), (4.4), (4.5) and (4.6) have, accordingly, been developed to analyze stationary points and the local, global stability of those points. The basic reproduction number is also discussed in the theorem-(4.1). As can be seen in this paper, when the value of the basic reproduction number (defined for the axial stationary point) is less than one, the pest population dies out after a certain amount of time. However, if the value of the basic reproduction number is higher than one and the death rate of the infected pest is 0.3, the graph of the model reaches the endemic state, and for the same $\mathbb{R}_0$. Also, it has been noted that the graph quickly stabilizes when the value of the fractional parameter drops.

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