SOME FIXED POINT THEOREMS OF RATIONAL TYPE CONTRACTION ON B-MULTIPLICATIVE METRIC SPACES

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Article History: Received: 17.02.2023 **Revised:** 06.03.2023 **Accepted:** 17.05.2023

Abstract

Our paper is ardent to prove fixed point theorems in the axioms of $b-$ multiplicative metric spaces for rational type contraction conditions. The presented theorem extends the research results of Merdaci Seddik1 and Hamaizia Taieb [6] and broadens a number of well-known findings in the setting of b-multiplicative metric spaces. Subsequently, we use our main results to find out the proximate solution of the Fredholm integral equation.

Keywords: b-multiplicative metric spaces ($\mathscr{b}-\mathscr{mms}$), Fixed point, Common Fixed point ($\mathcal{CFP}$), Cauchy sequence, Rational contractive condition.

Mathematical Subject Classification: Primary 47H10; Secondary 54H25

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DOI: 10.31838/ecb/2023.12.s2.279

1. Introduction

Grossman and Katz devise a multiplicative (Non-newtonian) calculus by surrogate division and multiplication for subtraction and addition, respectively. Bashirov et al. exemplify multiplicative metric by utilizing the theories of Grossman and Katz. Ozavsar and Cevikel studied multiplicative metric spaces and established certain fixed point theorems for multiplicative metric spaces contraction mappings. Abbas et al. made a significant contribution in this field with their investigation of common fixed point results of generalized rational type cyclic mappings. Czerwik coined the term "b-metric space," which is a generalization of the term "metric space". The interesting concept of b-multiplicative metric space was instigate by Muhammad Usman Ali.

II. Preliminaries

Definition 2.1

Let $\aleph$ be a non-empty set and $\mathfrak{s} \geq 1$ be a real number. A function $\upsilon: \aleph \times \aleph \to \mathbb{R}^+$ is a b −multiplicative metric if

i) $\upsilon(\mathfrak{u}, \mathfrak{v}) > 1$ for all $\mathfrak{u}, \mathfrak{v} \in \aleph$

ii) $\upsilon(\mathfrak{u}, \mathfrak{v}) = 1$ iff $\mathfrak{u} = \mathfrak{v}$

iii) $\upsilon(\mathfrak{u}, \mathfrak{v}) = \upsilon(\mathfrak{v}, \mathfrak{u})$ for all $\mathfrak{u}, \mathfrak{v} \in \aleph$

iv) $\upsilon(\mathfrak{u}, \mathfrak{z}) \leq [\upsilon(\mathfrak{u}, \mathfrak{v}).\upsilon(\mathfrak{v}, \mathfrak{z})]^{\mathfrak{s}}$ for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{z} \in \aleph$

The triplet $(\aleph, \upsilon, \mathfrak{s})$ is called b-multiplicative metric space.

Example 2.1

Let $\aleph = [0, \infty)$. Define a mapping $\upsilon: \aleph \times \aleph \to [1, \infty)$

$\upsilon(\mathfrak{u}, \mathfrak{v}) = a^{(\mathfrak{u}-\mathfrak{v})^2}$, for all $\mathfrak{u}, \mathfrak{v} \in \aleph$

where $a > 1$ is any fixed real number. Then for each a, υ is b −multiplicative metric on $\aleph$ with $\mathfrak{s} = 2$. Note that υ is not multiplicative metric on $\aleph$.

Definition 2.2

Let $(\aleph, \upsilon, \mathfrak{s})$ be a b −multiplicative metric space.

(i) A sequence $\{\mathfrak{u}_n\}$ is convergent iff there exist $\mathfrak{u} \in \aleph$ such that

$\upsilon(\mathfrak{u}_n, \mathfrak{u}) \to 1$ as $n \to +\infty$.

(ii) A sequence $\{\mathfrak{u}_n\}$ is called b −multiplicative Cauchy iff

$\upsilon(\mathfrak{u}_m, \mathfrak{u}_n) \to 1$ as $n, m \to \infty$.

iii) A b −multiplicative metric space $(\aleph, \upsilon)$ is said to be complete if every multiplicative Cauchy sequence in $\aleph$ is convergent to some $\mathfrak{v} \in \aleph$.

2. Main results

Lemma 3.1

Let $(\aleph, \upsilon, \mathfrak{s})$ be a $\mathscr{b}-\mathscr{mms}$ with coefficient $\mathfrak{s} \geq 1$ and $\mathcal{A}: \aleph \to \aleph$ be a mapping. Suppose that $\{\mathfrak{u}_n\}$ is a sequence in $\aleph$ induced by $\mathfrak{u}_{n+1} = \mathcal{A}\mathfrak{u}_n$ such that

$$\upsilon(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)^{\lambda}, \quad \text{for all } n \in \mathbb{N} \qquad \ldots(*)$$

where $\lambda \in [0,1)$ is a constant. Then $\{\mathfrak{u}_n\}$ is a Cauchy sequence.

Theorem 3.1:

Let $(\aleph, \upsilon, \mathfrak{s})$ be a complete $\mathscr{b}-\mathscr{mms}$ with a coefficient $\mathfrak{s} \geq 1$, and $\mathcal{A}, \mathcal{B} : \aleph \to \aleph$ be mappings on $\aleph$ satisfying the condition

$$\upsilon(\mathcal{A}\mathfrak{u}, \mathcal{B}\mathfrak{v}) \leq \left[\upsilon(\mathfrak{u}, \mathfrak{v})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u},\mathcal{A}\mathfrak{u})\upsilon(\mathfrak{u},\mathcal{B}\mathfrak{v})\upsilon(\mathfrak{v},\mathcal{B}\mathfrak{v})\upsilon(\mathfrak{v},\mathcal{A}\mathfrak{u})}{\upsilon(\mathfrak{u},\mathcal{B}\mathfrak{v})+\upsilon(\mathfrak{v},\mathcal{A}\mathfrak{u})}\right]^{\alpha_2}\right] \qquad \ldots(1)$$

for all $\mathfrak{u}, \mathfrak{v}$ in $\aleph$ and $\alpha_1, \alpha_2 \geq 0$, $\upsilon(\mathfrak{u}, \mathcal{B}\mathfrak{v}) + \upsilon(\mathfrak{v}, \mathcal{A}\mathfrak{u}) \neq 0$ with $\alpha_1 + \alpha_2 < 1$. Then $\mathcal{A}$ and $\mathcal{B}$ has a unique $\mathcal{CFP}$.

Proof:

Fix $\mathfrak{u}_0 \in \aleph$. Define a sequence $\{\mathfrak{u}_n\}$ in $\aleph$ such that

$$\mathfrak{u}_{2n+1} = \mathcal{A}\mathfrak{u}_{2n},\ \mathfrak{u}_{2n+2} = \mathcal{B}\mathfrak{u}_{2n+1}, \text{ for all } n \in \mathbb{N}. \qquad \ldots(2)$$

Suppose that there is some $n \in \mathbb{N}$ such that $\mathfrak{u}_n = \mathfrak{u}_{n+1}$. If $n = 2i$, then $\mathfrak{u}_{2i} = \mathfrak{u}_{2i+1}$ and

From (1) with $\mathfrak{u} = \mathfrak{u}_{2i}$ and $\mathfrak{v} = \mathfrak{u}_{2i+1}$,

$$\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+2}) = \upsilon(\mathcal{A}\mathfrak{u}_{2i}, \mathcal{B}\mathfrak{u}_{2i+1})$$

$$\leq \left[\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u}_{2i}, \mathcal{A}\mathfrak{u}_{2i})\upsilon(\mathfrak{u}_{2i}, \mathcal{B}\mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i+1}, \mathcal{B}\mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i+1}, \mathcal{B}\mathfrak{u}_{2i})}{\upsilon(\mathfrak{u}_{2i}, \mathcal{B}\mathfrak{u}_{2i+1}) + \upsilon(\mathfrak{u}_{2i+1}, \mathcal{B}\mathfrak{u}_{2i})}\right]^{\alpha_2}\right]$$

$$= \left[\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+2})\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+2})\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+1})}{\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+2}) + \upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+1})}\right]^{\alpha_2}\right]$$

$$= 1.$$

Therefore $\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+2}) = 1$ which implies $\mathfrak{u}_{2i+1}, = \mathfrak{u}_{2i+2}$.

Thus $$\mathfrak{u}_{2i} = \mathfrak{u}_{2i+1} = \mathfrak{u}_{2i+2}.$$

By (2), $\mathcal{A}\mathfrak{u}_{2i} = \mathfrak{u}_{2i} = \mathcal{B}\mathfrak{u}_{2i}$, $\mathfrak{u}_{2i}$ is a $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$.
If $n = 2i + 1$, then using the same argument as in the case $\mathfrak{u}_{2i} = \mathfrak{u}_{2i+1}$, it can be shown that $\mathfrak{u}_{2i+1}$ is a $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$.
Suppose, $\mathfrak{u}_n \neq \mathfrak{u}_{n+1}$ for all $n \in \mathbb{N}$.
Step 1:
Case 1: $n = 2i + 1, i \in \mathbb{N}$
Use the equation (1), with $\mathfrak{u} = \mathfrak{u}_{2i}$ and $\mathfrak{v} = \mathfrak{u}_{2i+1}$

$$\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+2}) = \upsilon(\mathcal{A}\mathfrak{u}_{2i}, \mathcal{B}\mathfrak{u}_{2i+1})$$
$$\leq \left[\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u}_{2i}, \mathcal{A}\mathfrak{u}_{2i})\upsilon(\mathfrak{u}_{2i}, \mathcal{B}\mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i+1}, \mathcal{B}\mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i+1}, \mathcal{B}\mathfrak{u}_{2i})}{\upsilon(\mathfrak{u}_{2i}, \mathcal{B}\mathfrak{u}_{2i+1}) + \upsilon(\mathfrak{u}_{2i+1}, \mathcal{B}\mathfrak{u}_{2i})}\right]^{\alpha_2}\right]$$
$$\leq \left[\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+2})\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+2})\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+1})}{\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+2}) + \upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+1})}\right]^{\alpha_2}\right]$$
$$= \left[\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+2})\upsilon(\mathfrak{u}_{2i+1}, \mathfrak{u}_{2i+2})}{\upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+2}) + 1}\right]^{\alpha_2}\right]$$
$$= \upsilon(\mathfrak{u}_{2i}, \mathfrak{u}_{2i+1})^{\alpha_1+\alpha_2}.$$

In general,

$$\upsilon(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)^{\alpha_1+\alpha_2}, \quad n = 2i + 1, i \in \mathbb{N} \quad \ldots (3)$$

Case 2: $n = 2i, i \in \mathbb{N}$.
Likewise, For $n = 2i$

$$\upsilon(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)^{\alpha_1+\alpha_2}, \quad n = 2i, i \in \mathbb{N} \quad \ldots (4)$$

Utilizing the equation (3) and (4),

$$\upsilon(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)^{\alpha_1+\alpha_2}, \text{ for all } n \in \mathbb{N}. \quad \ldots (5)$$

$\because \alpha_1 + \alpha_2 < 1$.
By Lemma (3.1), Thus $\{\mathfrak{u}_n\}$ is a Cauchy sequence in $(\aleph, \upsilon, \mathfrak{s})$. Since $(\aleph, \upsilon, \mathfrak{s})$ is a complete $\mathscr{b} - \mathscr{mms}$, $\{\mathfrak{u}_n\}$ converges to some $z \in \aleph$ as $n \to +\infty$.
Step 2:
To prove that $\mathcal{A}z = \mathcal{B}z = z$.
Using the triangular inequality and (1), $\upsilon(z, \mathcal{A}z) \leq [\upsilon(z, \mathfrak{u}_{2n+2}).\upsilon(\mathfrak{u}_{2n+2}, \mathcal{A}z)]^{\mathfrak{s}}$

$$= \upsilon(z, \mathfrak{u}_{2n+2})^{\mathfrak{s}}.\upsilon(\mathcal{A}z, \mathcal{B}\mathfrak{u}_{2n+1})^{\mathfrak{s}}$$
$$\leq \upsilon(z, \mathfrak{u}_{2n+2})^{\mathfrak{s}}.\left[\upsilon(z, \mathfrak{u}_{2n+2})^{\mathfrak{s}\alpha_1}.\left[\frac{\upsilon(z, \mathcal{A}z)\upsilon(z, \mathcal{B}\mathfrak{u}_{2n+1})\upsilon(\mathfrak{u}_{2n+1}, \mathcal{B}\mathfrak{u}_{2n+1})\upsilon(\mathfrak{u}_{2n+1}, \mathcal{A}z)}{\upsilon(z, \mathcal{B}\mathfrak{u}_{2n+1}) + \upsilon(\mathfrak{u}_{2n+1}, \mathcal{A}z)}\right]^{\mathfrak{s}\alpha_2}\right]$$
$$= \upsilon(z, \mathfrak{u}_{2n+2})^{\mathfrak{s}}.\left[\upsilon(z, \mathfrak{u}_{2n+2})^{\mathfrak{s}\alpha_1}.\left[\frac{\upsilon(z, \mathcal{A}z)\upsilon(z, \mathfrak{u}_{2n+2})\upsilon(\mathfrak{u}_{2n+1}, \mathfrak{u}_{2n+2})\upsilon(\mathfrak{u}_{2n+1}, \mathcal{A}z)}{\upsilon(z, \mathfrak{u}_{2n+2}) + \upsilon(\mathfrak{u}_{2n+1}, \mathcal{A}z)}\right]^{\mathfrak{s}\alpha_2}\right]$$

Taking the limit as $n \to \infty$, obtain $\upsilon(z, \mathcal{A}z) \leq 1$, Hence $\upsilon(z, \mathcal{A}z) = 1$ which implies $\mathcal{A}z = z$.
Similarly, we obtain $\mathcal{B}z = z$, Thus z is a $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$.
Step 3:
To determine, $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point.
Assume z and w are different $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$,
By the equation (1),

$$\upsilon(z, w) = \upsilon(\mathcal{A}z, \mathcal{B}w)$$
$$\leq \upsilon(z, w)^{\alpha_1}.\left[\frac{\upsilon(z, \mathcal{A}z).\upsilon(z, \mathcal{B}w)\upsilon(w, \mathcal{B}w)\upsilon(w, \mathcal{B}w)}{\upsilon(z, \mathcal{B}w) + \upsilon(w, \mathcal{A}z)}\right]^{\alpha_2}$$
$$= \upsilon(z, w)^{\alpha_1}$$

Since $\alpha_1 < 1$, we have $\upsilon(z, w) = 1$.
Thus, $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point in $\aleph$.

Theorem 3.2:
Let $(\aleph, \upsilon, \mathfrak{s})$ be a complete $\mathscr{b} - \mathscr{mms}$ with coefficient $\mathfrak{s} \geq 1$, and $\mathcal{A}, \mathcal{B}: \aleph \to \aleph$ be mappings on $\aleph$, The following conditions are fulfilled

$$\upsilon(\mathcal{A}\mathfrak{u}, \mathcal{B}\mathfrak{v}) \leq \left[\upsilon(\mathfrak{u}, \mathfrak{v})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{v}, \mathcal{B}\mathfrak{v})[1+\upsilon(\mathfrak{u}, \mathcal{A}\mathfrak{u})]}{1+\upsilon(\mathfrak{u}, \mathfrak{v})}\right]^{\alpha_2}.\left[\frac{\upsilon(\mathfrak{v}, \mathcal{B}\mathfrak{v})+\upsilon(\mathfrak{v}, \mathcal{A}\mathfrak{u})}{1+\upsilon(\mathfrak{v}, \mathcal{B}\mathfrak{v})\upsilon(\mathfrak{v}, \mathcal{B}\mathfrak{u})}\right]^{\alpha_3}\right], \quad \ldots (6)$$

For all $\mathfrak{u}, \mathfrak{v} \in \aleph$ where $\alpha_1, \alpha_2, \alpha_3 \geq 0$, $\mathfrak{s}(\alpha_1 + \alpha_2 + \alpha_3) < 1$. Then $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point.

Proof:
Any $\mathfrak{u}_0$ in $\aleph$, A sequence $\{\mathfrak{u}_n\}$ in $\aleph$ such that

$$\mathfrak{u}_{2n+1} = \mathcal{A}\mathfrak{u}_{2n},\ \mathfrak{u}_{2n+2} = \mathcal{B}\mathfrak{u}_{2n+1}, \text{ for all } n \in \mathbb{N}. \quad \ldots (7)$$

Suppose that there is some $n \in \mathbb{N}$ such that $u_n = u_{n+1}$.
If $n = 2i$, then $u_{2i} = u_{2i+1}$,
Use the equation (6) with $u = u_{2i}$ and $v = u_{2i+1}$,

$$\upsilon(u_{2i+1}, u_{2i+2}) = \upsilon(\mathcal{A}u_{2k}, \mathcal{B}u_{2k+1})$$
$$\leq \upsilon(u_{2i}, u_{2i+1})^{\alpha_1} . \left[\frac{\upsilon(u_{2i+1}, \mathcal{B}u_{2i+1})[1 + \upsilon(u_{2i}, \mathcal{A}u_{2i})]}{1 + \upsilon(u_{2i}, u_{2i+1})}\right]^{\alpha_2} . \left[\frac{\upsilon(u_{2i+1}, \mathcal{B}u_{2i+1}) + \upsilon(u_{2i+1}, \mathcal{A}u_{2i})}{1 + \upsilon(u_{2i+1}, \mathcal{B}u_{2i+1})\upsilon(u_{2i+1}, \mathcal{B}u_i)}\right]^{\alpha_3}$$
$$= \upsilon(u_{2i}, u_{2i+1})^{\alpha_1} . \left[\frac{\upsilon(u_{2i+1}, u_{2i+2})[1 + \upsilon(u_{2i}, u_{2i+1})]}{1 + \upsilon(u_{2i}, u_{2i+1})}\right]^{\alpha_2} . \left[\frac{\upsilon(u_{2i+1}, u_{2i+2}) + \upsilon(u_{2i+1}, u_{2i+1})}{1 + \upsilon(u_{2i+1}, u_{2i+2})\upsilon(u_{2i+1}, u_{2i+1})}\right]^{\alpha_3}$$
$$\upsilon(u_{2i+1}, u_{2i+2})^{1-\alpha_3} \leq 1.$$

Since $0 \leq \alpha_3 < 1$, $\upsilon(u_{2i+1}, u_{2i+2}) = 1$. Hence $u_{2i+1} = u_{2i+2}$. Thus, $u_{2i} = u_{2i+1} = u_{2i+2}$.
We find by the equation (7), $u_{2i} = \mathcal{A}u_{2i} = \mathcal{B}u_{2i}$, i.e., u_{2i} is a $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$.
If $n = 2i + 1$, then using the same arguments as in the case $u_{2i} = u_{2i+1}$, it can be shown that u_{2i+1} is a $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$.
Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$.
Step 1:
Case 1:If $n = 2i + 1, i \in \mathbb{N}$.
Use the equation (6), $u = u_{2i}$ and $v = u_{2i+1}$

$$\upsilon(u_{2i+1}, u_{2i+2}) = \upsilon(\mathcal{A}u_{2i}, \mathcal{B}u_{2i+1})$$
$$\leq \upsilon(u_{2i}, u_{2i+1})^{\alpha_1} . \left[\frac{\upsilon(u_{2i+1}, \mathcal{B}u_{2i+1})[1 + \upsilon(u_{2i}, \mathcal{A}u_{2i})]}{1 + \upsilon(u_{2i}, u_{2i+1})}\right]^{\alpha_2} . \left[\frac{\upsilon(u_{2i+1}, \mathcal{B}u_{2i+1}) + \upsilon(u_{2i+1}, \mathcal{A}u_{2i})}{1 + \upsilon(u_{2i+1}, \mathcal{B}u_{2i+1})\upsilon(u_{2i+1}, \mathcal{B}u_{2i})}\right]^{\alpha_3}$$
$$= \upsilon(u_{2i}, u_{2i+1})^{\alpha_1} . \left[\frac{\upsilon(u_{2i+1}, u_{2i+2})[1 + \upsilon(u_{2i}, u_{2i+1})]}{1 + \upsilon(u_{2i}, u_{2i+1})}\right]^{\alpha_2} . \left[\frac{\upsilon(u_{2i+1}, u_{2i+2}) + \upsilon(u_{2i+1}, u_{2i+1})}{1 + \upsilon(u_{2i+1}, u_{2i+2})\upsilon(u_{2i+1}, u_{2i+1})}\right]^{\alpha_3}$$
$$= \upsilon(u_{2i}, u_{2i+1})^{\frac{\alpha_1}{1-(\alpha_1+\alpha_3)}}$$

Thus $\upsilon(u_n, u_{n+1}) \leq \upsilon(u_{n-1}, u_n)^{\frac{\alpha_1}{1-(\alpha_1+\alpha_3)}}, \ n = 2i + 1, i \in \mathbb{N}$ …(8)
Case 2: If $n = 2i, i \in \mathbb{N}$.
Likewise, For $n = 2i$,

$\upsilon(u_n, u_{n+1}) \leq \upsilon(u_{n-1}, u_n)^{\frac{\alpha_1}{1-(\alpha_1+\alpha_3)}}, \ n = 2i, i \in \mathbb{N}$ …(9)
Utilizing (8) and (9),

$\upsilon(u_n, u_{n+1}) \leq \upsilon(u_{n-1}, u_n)^{\frac{\alpha_1}{1-(\alpha_1+\alpha_3)}}$, for all $n \in \mathbb{N}$, …(10)
Where $h = \frac{\alpha_1}{1-(\alpha_1+\alpha_3)}$ with $h < \frac{1}{\mathfrak{s}} \leq 1$, $\mathfrak{s}(\alpha_1 + \alpha_2 + \alpha_3) < 1$.
The Lemma (3.1), we say that $\{u_n\}$ is a Cauchy sequence in $(\aleph, \upsilon)$. Since $(\aleph, \upsilon)$ is a complete $b - mms$, $\{u_n\}$ converges to some $z \in \aleph$ as $n \to +\infty$.
Step 2:
To demonstrate $\mathcal{A}z = \mathcal{B}z = z$.
By using the triangular inequality and (6),

$$\upsilon(z, \mathcal{A}z) \leq [\upsilon(z, u_{2n+2}).\upsilon(u_{2n+2}, \mathcal{A}z)]^{\mathfrak{s}}$$
$$= \upsilon(z, u_{2n+2})^{\mathfrak{s}}.\upsilon(\mathcal{A}z, \mathcal{B}u_{2n+1})^{\mathfrak{s}}$$
$$\leq \upsilon(z, u_{2n+2})^{\mathfrak{s}}.\upsilon(z, u_{2n+1})^{\mathfrak{s}\alpha_1} . \left[\frac{\upsilon(u_{2n+1}, \mathcal{B}u_{2n+1})[1 + \upsilon(z, \mathcal{A}z)]}{1 + \upsilon(z, u_{2n+1})}\right]^{\mathfrak{s}\alpha_2} . \left[\frac{\upsilon(u_{2n+1}, \mathcal{B}u_{2n+1}) + \upsilon(u_{2n+1}, \mathcal{A}z)}{1 + \upsilon(u_{2n+1}, \mathcal{B}u_{2n+1})\upsilon(u_{2n+1}, \mathcal{A}z)}\right]^{\mathfrak{s}\alpha_3}.$$

Taking the limit as $n \to +\infty$, $\upsilon(z, \mathcal{A}z) \leq \upsilon(z, \mathcal{A}z)^{\mathfrak{s}\alpha_3}$
Since $\mathfrak{s}\alpha_3 < 1$, hence $\upsilon(z, \mathcal{A}z)=1$, Thus $\mathcal{A}z = z$.
Likewise, we obtain $\upsilon(z, \mathcal{B}z) \leq \upsilon(z, \mathcal{B}z)^{\mathfrak{s}(\alpha_2+\alpha_3)}$.
$\because \mathfrak{s}(\alpha_2 + \alpha_3) < 1, \upsilon(z, \mathcal{B}z) = 1$, Thus $\mathcal{B}z = z$
Thus z is a $\mathcal{CFP}$ of $\mathcal{A}$ and $\mathcal{B}$.
Step 3:
To determine $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point.
Say z and w are different common fixed points of $\mathcal{A}$ and $\mathcal{B}$,
As a result (6), $\upsilon(z, w) = \upsilon(\mathcal{A}z, \mathcal{B}w)$

$$\leq \left[\upsilon(z, w)^{\alpha_1} . \left[\frac{\upsilon(w, \mathcal{B}w)[1 + \upsilon(z, \mathcal{A}z)]}{1 + \upsilon(z, w)}\right]^{\alpha_2} . \left[\frac{\upsilon(w, \mathcal{B}w)\upsilon(w, \boldsymbol{\mathcal{A}z})}{1 + \upsilon(w, \mathcal{B}w)\upsilon(w, \mathcal{B}z)}\right]^{\alpha_3}\right]$$
$$= \upsilon(z, w)^{(\alpha_1+\alpha_3)}$$

Since $0 < (\alpha_1 + \alpha_3) < 1, \upsilon(z, w) = 1$

Thus, we prove that $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point in $\aleph$.

Theorem 3.3:

Let $(\aleph, \upsilon, \mathfrak{s})$ be a complete $\mathcal{b}-mms$ with a coefficient $\mathfrak{s} \geq 1$, $f: \aleph \to \aleph$ be a mapping on $\aleph$. Suppose that $\alpha_1, \alpha_2, \alpha_3$ are nonnegative reals with $\alpha_1 + \alpha_3 < 1, \frac{\alpha_1+\alpha_2}{\mathfrak{s}-\alpha_3} < 1$ such that the inequality

$\upsilon(f\mathfrak{u}, f\mathfrak{v})^{\mathfrak{s}} \leq \upsilon(\mathfrak{u}, \mathfrak{v})^{\alpha_1}. \left[\frac{\upsilon(\mathfrak{u}, f\mathfrak{v})\upsilon(y, f\mathfrak{v})}{1+\upsilon(f\mathfrak{u}, f\mathfrak{v})}\right]^{\alpha_2}. \upsilon(f\mathfrak{u}, f\mathfrak{v})^{\alpha_3}$, …(11)

holds for each $\mathfrak{u}, \mathfrak{v} \in \aleph$. Then f has a unique fixed point.

Proof:

Let $\mathfrak{u}_0$ be arbitrary in $\aleph$. We define a sequence $\{\mathfrak{u}_n\}$ in $\aleph$ such that

$\mathfrak{u}_{n+1} = f\mathfrak{u}_n$, for all $n \in \mathbb{N}$ …(12)

From the condition (11) with $\mathfrak{u} = \mathfrak{u}_n$ and $\mathfrak{v} = \mathfrak{u}_{n-1}$, Therefore

$\upsilon(\mathfrak{u}_{n+1}, \mathfrak{u}_n)^{\mathfrak{s}} = \upsilon(f\mathfrak{u}_n, f\mathfrak{u}_{n-1})^{\mathfrak{s}}$

$\leq \upsilon(\mathfrak{u}_n, \mathfrak{u}_{n-1})^{\alpha_1}. \left[\frac{\upsilon(\mathfrak{u}_n, f\mathfrak{u}_n)\upsilon(\mathfrak{u}_{n-1}, f\mathfrak{u}_{n-1})}{1+\upsilon(f\mathfrak{u}_n, f\mathfrak{u}_{n-1})}\right]^{\alpha_2}. \upsilon(f\mathfrak{u}_n, f\mathfrak{u}_{n-1})^{\alpha_3}$

$= \upsilon(\mathfrak{u}_n, \mathfrak{u}_{n-1})^{\alpha_1}. \left[\frac{\upsilon(\mathfrak{u}_n, \mathfrak{u}_{n+1})\upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}{1+\upsilon(\mathfrak{u}_{n+1}, \mathfrak{u}_n)}\right]^{\alpha_2}. \upsilon(\mathfrak{u}_{n+1}, \mathfrak{u}_n)^{\alpha_3}$

$\leq \upsilon(\mathfrak{u}_n, \mathfrak{u}_{n-1})^{\alpha_1}. \left[\frac{\upsilon(\mathfrak{u}_n, \mathfrak{u}_{n+1})\upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)}{1+\upsilon(\mathfrak{u}_{n+1}, \mathfrak{u}_n)}\right]^{\alpha_2}. \upsilon(\mathfrak{u}_{n+1}, \mathfrak{u}_n)^{\alpha_3}$

$\upsilon(\mathfrak{u}_{n+1}, \mathfrak{u}_n) \leq \upsilon(\mathfrak{u}_{n-1}, \mathfrak{u}_n)^{\frac{\alpha_1+\alpha_2}{\mathfrak{s}-\alpha_3}}$

By using the Lemma (3.1) we say that $\{\mathfrak{u}_n\}$ is a Cauchy sequence in $(\aleph, \upsilon)$. Since $(\aleph, \upsilon)$ is a complete $\mathcal{b}-mms$, then $\{\mathfrak{u}_n\}$ converges to some $z \in \aleph$ as $n \to \infty$.

To show $fz = z$

Again by triangle inequality and (11),

$\upsilon(z, fz) \leq [\upsilon(z, \mathfrak{u}_{n+1}). (\upsilon(\mathfrak{u}_{n+1}, fz)]^{\mathfrak{s}}$

$= \upsilon(z, \mathfrak{u}_{n+1})^{\mathfrak{s}}\upsilon(fz, f\mathfrak{u}_n)^{\mathfrak{s}}$

$\leq \upsilon(z, \mathfrak{u}_{n+1})^{\mathfrak{s}}. \upsilon(z, \mathfrak{u}_n)^{\mathfrak{s}\alpha_1}. \left[\frac{\upsilon(\mathfrak{u}_n, f\mathfrak{u}_n)\upsilon(z, fz)}{1+\upsilon(fx_n, fz)}\right]^{\mathfrak{s}\alpha_2}. \upsilon(fz, f\mathfrak{u}_n)^{\mathfrak{s}\alpha_3}$

Taking the limit as $n \to \infty$, we obtain that

$$\upsilon(z, fz)^{1-\alpha_3} \leq 1,$$

Since $0 < \alpha_3 < 1$, then

$\upsilon(z, fz) \leq 1$.

Which is a contradiction, so $\upsilon(z, fz) = 1$. Hence $fz = z$, thus z is a fixed point of f. We will prove that f have a unique fixed point. Suppose z and w are different fixed points of f, then from (11), it follows that

$\upsilon(z, w)^{\mathfrak{s}} = \upsilon(fz, fw)^{\mathfrak{s}} \leq \upsilon(z, w)^{\alpha_1}. \left[\frac{\upsilon(z, fz)\upsilon(w, fw)}{1+\upsilon(fz, fw)}\right]^{\alpha_2}. \upsilon(fz, fw)^{\alpha_3}$,

$= \upsilon(z, w)^{\alpha_1}. \left[\frac{\upsilon(z, z)\upsilon(w, w)}{1+\upsilon(z, w)}\right]^{\alpha_2}. \upsilon(z, w)^{\alpha_3}$

$= \upsilon(z, w)^{\alpha_1+\alpha_3}$.

Since $\alpha_1 + \alpha_3$ is non negative reals with $\alpha_1 + \alpha_3 < 1$, then we have $\upsilon(z, w) = 1$.

Thus, f have a unique fixed point in $\aleph$.

Application:

We explore the existence and uniqueness of the solution of a system of multiplicative integral equations. The integral equation as

$\mathfrak{u}(t) = \int_a^b K(t, s, \mathfrak{u}(s))^{ds}$, $s, t \in [a, b]$ …(4.1)

Where $a, b \in \mathbb{R}$ and $K: [a, b] \times [a, b] \times \mathbb{R}$. The purpose of this section is to give an existence theorem for a solution of Theorem (4.1) using theorem 3.3

Consider the space $\aleph = C[a, b]$ of real continuous functions defined on $[a, b]$. Endowed with b −multiplicative metric

$\upsilon(\mathfrak{u}, \mathfrak{v}) = \begin{cases} sup_{t\in[a,b]} \left|\frac{\mathfrak{u}(t)}{\mathfrak{v}(t)}\right|^2 \ if \ \frac{\mathfrak{u}(t)}{\mathfrak{v}(t)} > 1 \\ sup_{t\in[a,b]} \left|\frac{\mathfrak{v}(t)}{\mathfrak{u}(t)}\right|^2 \ if \ \frac{\mathfrak{u}(t)}{\mathfrak{v}(t)} < 1 \end{cases}$ is a complete b −multiplicative metric space

Theorem 4.1:

Assume that

i) For each $t, s \in [a, b]$ and $\mathfrak{u}, \mathfrak{v} \in \aleph$, there exists a constant $\eta > 0$ such that

$$\left|\frac{K\big(t,s,\mathfrak{u}(s)\big)}{K\big(t,s,\mathfrak{v}(s)\big)}\right|^2 \le \left(\left|\frac{\mathfrak{u}(s)}{\mathfrak{v}(s)}\right|\right)^{\eta},$$

ii) the constant η is such that $\eta < \frac{1}{q(b-a)}, 1 < q < \infty$,
iii) K: $[a,b] \times [a,b] \times \mathbb{R}$ is continuous.
Then the system (4.1) have a unique common solution in $\aleph$.
Proof:
Let F: $\aleph \times \aleph$ defined as

$$F\mathfrak{u}(t) = \int_a^b K(t,s,\mathfrak{u}(s))^{ds},\ s,t \in [a,b]$$

Since K is continuous, f is continuous

$$\left|\frac{F\mathfrak{u}(t)}{F\mathfrak{v}(t)}\right|^2 \le \left(\int_a^b \left|\frac{K\big(t,s,\mathfrak{u}(s)\big)}{K\big(t,s,\mathfrak{v}(s)\big)}\right|^{ds}\right)^2$$

$$\le \left(\int_a^b \left(\left|\frac{\mathfrak{u}(s)}{\mathfrak{v}(s)}\right|^{\eta}\right)^{ds}\right)^2$$

$$\le \left(\int_a^b \left(\upsilon(\mathfrak{u},\mathfrak{v})^{\frac{\eta}{2}}\right)^{ds}\right)^2$$

$$= \left(\left(\upsilon(\mathfrak{u},\mathfrak{v})^{b-a}\right)^{\frac{\eta}{2}}\right)^2$$

$= \upsilon(\mathfrak{u},\mathfrak{v})^{\eta(b-a)}$ for each $t \in [a,b]$.

$$< \upsilon(\mathfrak{u},\mathfrak{v})^{\frac{1}{q}}$$

$\upsilon(f\mathfrak{u},f\mathfrak{v})^{\mathfrak{s}} \le [\upsilon(\mathfrak{u},\mathfrak{v})^{\alpha_1}.\left[\frac{\upsilon(\mathfrak{u},f\mathfrak{v})\upsilon(\mathfrak{v},f\mathfrak{v})}{1+\upsilon(f\mathfrak{u},f\mathfrak{v})}\right]^{\alpha_2}.\upsilon(f\mathfrak{u},f\mathfrak{v})^{\alpha_3}]^{\frac{1}{q}}$ where $1 \le \mathfrak{s} < q < \infty$.
For all $\mathfrak{u},\mathfrak{v} \in \aleph$. Consequently, all the hypothesis of Theorem 3.3 hold. Then f have a unique common fixed point and the system (4.1) have a unique common solution.

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