ESTIMATION OF MEASURES GENERATED BY NEW F-DIVERGENCE MEASURE VIA JENSEN'S INEQUALITY ON TIME SCALE

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Abstract:
Divergence measure is the tool and solve the problem which can be use in the filed information theoretic problems. The reason of this paper is to discover a few time scale disparities for distinctive divergence measure by use of weighted Jensen's inequality. These comes about gives unused disparities in h-discrete calculus and quantum calculus and also gives the lower bounds of divergence measure.

Keywords: Information theory, Convex function, f-divergence measure, Time scale, Continuous entropy, Jensen's inequality.

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1. Introduction

Divergence measures have important role in the information theory and statistics. Basically divergence measure is the distance between two probability distributions. Jain et al. (2012 and 2013) presented new-f-divergence measure and its properties which is given as takes after.

Definition1.1 [8]: Let $f: R_+ \to (0.5, \infty)$ is a convex function and also let $U = \{u_1, u_2, u_3........u_n\}$ and $V = \{v_1, v_2, v_3,............v_n\}$ be such that $\sum_{i=1}^{n} u_i = 1$ and $\sum_{i=1}^{n} v_i = 1$ then the divergence measure known as new-f-divergence measure.

$$S_f(U,V) = \sum_{i=1}^{n} V_i f\left(\frac{U_i + V_i}{2V_i}\right)$$

The requirements of the function f is.

$$f(0) = \underset{\eta \longrightarrow 0^+}{Lim} f(\eta)\ ; \quad 0f(0) = 0;\ 0f\left(\frac{a}{0}\right) := Lim_{\eta \longrightarrow 0^+} \eta f\left(\frac{a}{\eta}\right),\ a>0$$

By the use of new f-divergence we can derive other well-known divergence measure which is discussed in [6].

Definition 1.2 [2]: let T^0 be a time scale and $\varepsilon \in T^0$ at that point

$$\alpha(\varepsilon) := \inf\left\{q \in T^0 : q > \varepsilon\right\} \text{ and } \beta(\varepsilon) := \sup\left\{q \in T^0 : q < \varepsilon\right\}$$

Definition 1.3 [2]: Consider the set T^0 as follows:

$$T^{0^i} = \begin{cases} T^0 \backslash \beta(\sup T^0), ^{-} \sup T^0 & if \sup T^0 < \infty \\ T^0 & if \sup T^0 = \infty \end{cases}$$

Where
T^0 = time scale,
$X : T^0 \to R$ is a function,

Definition 1.4 [2]: consider a function $X : T^0 \longrightarrow R$ and $\varepsilon \in T^{0^i}$ then defined $X^{\Delta}(\varepsilon)$ to the number with $\varepsilon > 0$, there is a neighbour N of ε such that

$$\left|x(\alpha(\varepsilon)) - x^{\Delta}(\varepsilon)(\alpha(\varepsilon) - q)\right| \le \eta\left|\beta(\varepsilon) - q\right| \quad \forall q \in N$$

For $T^0 = R$ (set of real number) then x^{Δ} becomes ordinary derivatives, and if $T^0 = Z$ (set of integer) then x^{Δ} turns into the forward difference operator $\Delta x(\varepsilon) = x(\varepsilon + 1) = x(\varepsilon)$, if $T^0 = p^{\bar{x}} = \{p^n : n \in z\} \cup \{0\}$ is called q difference operator with $p > 1$ then

$$x^{\Delta} = \frac{x(p\varepsilon) - x(\varepsilon)}{(p-1)\varepsilon},\ x^{\Delta}(0) = \underset{p \longrightarrow 0}{\lim} \frac{x(p) - x(0)}{p}$$

Theorem 1.1 [2]: if $X_0 \in T^0$, then rd-continuous function has an rd- triderivative. The function J defined by $J(\in) = \int_{X_0}^{X} j(\in)\Delta\in$ for $X_0 \in T^{0^k}$ is an anti-derivative of j.

Case-1: for the condition $T^0 = R$ (set of real number), then

$$\int_e^f j(\in)d\in=\int_e^f j(\in)\Delta\in$$

Case-2: for the condition $T^0=N$ (set of natural number), then

$$\int_e^f j(\in)d\in=\sum_{\in=e}^{f-1} j(\in)$$, where $e,f\in T^0$ with the condition $e\le f$.

Theorem 1.2 [2]: let $I\subset R$ and $v\in C_{rd}^0\left([e,f]_{T^0},R\right)$ is a positive function with $\int_e^f v(\in)\Delta\in>0,$ where $e,f\in T^0$ .if $J\in C(I,R)$ is convex and $g=C_{rd}^0\left([e,f]_{T^0},I\right)$, then

$$j\left(\frac{\int_e^f v(\in)g(\in)\Delta\in}{\int_e^f v(\in)\Delta\in}\right)\le\frac{\int_e^f v(\in)J\left(g(\in)\right)\Delta\in}{\int_e^f v(\in)\Delta\in} \tag{1.1}$$

Where J is the strictly convex function.

2. New f-divergence on the time scale

Let on the time scale T^0 the following set is rd-continuous

$$Q=\left\{r\in c_{rd}^0\left([e,f]_{T^0},(0,\infty)\right),r(\in)>0,\int_e^f r(\in)\Delta\in>0\right\}$$

Within the continuation, we expect that $r,r(\in)\in Q$ and exist the taking after integrand $R^0=\int_e^f r(\in)\Delta\in$ and $S^0=\int_e^f s(\in)\Delta\in$

i. New f- divergence measure

Here we can define the New f-divergence with the time scale which is characterized as follows:

$$S_f(U,V)=\int_e^f v(\in)f\left(\frac{u(\in)+v(\in)}{2v(\in)}\right)\Delta\in \tag{2.1}$$

When f is convex function on R^+.

Theorem 2.1: Let us consider $I\subset R$, the convex function $f\in C^0(I,R)$ then

$$vf\left(\frac{u+v}{2v}\right)\le S_f(u,v) \tag{2.2}$$

Where $S_f(u,v)$ is given by the equation (2.2).

Example 2.1 for the condition $T^0=R$, theorem (2.1) becomes.

Example 2.2 Take $T^0=h^0Z,h^0>0$ using theorem (2.1), we can obtain a lower bound of the new f-divergence with h^0 -discrete calculus

$$\sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0 f\left(\frac{\sum_{l_0=e/h^0}^{f/h^0-1} u\left(l_0h^0\right)h^0 + \sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0}{2\sum_{l_0=e/h^0}^{f/h^0-1} v\left(l^0h^0\right)h^0}\right) \le \sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0 f\left(\frac{u\left(l_0h^0\right)h^0 + v\left(l_0h^0\right)h^0}{2v\left(l_0h^0\right)h^0}\right)$$

Remark 2.1 Now take $h^0 = 1$ in the previous example and also put $e = 0, f = n$, $v(i) = v_j$ and $u(i) = u_j$ to get f-divergence measure.

$$\sum_{j=1}^{n} v_j f\left(\frac{\sum_{j=1}^{n} u_j + \sum_{j=1}^{n} v_j}{2\sum_{j=1}^{n} v_j}\right) \le S_f(u,v) \qquad (2.3)$$

Example 2.3 Take $T^0 = p^{N^0}$ $(p > 1)$ in the theorem (2.1) have a new lower bounds of the divergence

$$\sum_{l_0=1}^{n-1} p^{l_0+1} r\left(p^{l_0}\right) f\left(\frac{\sum_{l_0=0}^{n-1} p^{l_0+1} u\left(p^{l_0}\right) + \sum_{l_0=0}^{n-1} p^{l_0+1} v\left(p^{l_0}\right)}{2\sum_{l_0=0}^{n-1} p^{l_0+1} v\left(p^{l_0}\right)}\right) \le \sum_{l_0=0}^{n-1} p^{l_0+1} r\left(p^{l_0}\right) f\left(\frac{u\left(p^{l_0}\right) + v\left(p^{l_0}\right)}{2v\left(p^{l_0}\right)}\right) \qquad (2.4)$$

ii. Continuous entropy for new f-divergence measure

Let r is a positive density function which is defined T^0 and continuous arbitrary Υ then we get $\int_e^f v(\in)\Delta\in = 1$ whenever the integrals exists by Ansari **[3]**

$$h_b^0(\Upsilon) := \int_e^f v(\in)\log\frac{1}{v(\in)}\Delta\in \qquad (2.5)$$

Where $\overline{f} > 1$ becomes is the base log. The base of $\log \ge 1$.

Theorem 2.2: Let $(u, v \in C_{rd}^0[e, f]_{T^0}, R)$ be Δ - integrable and u is a positive PF ("probability function") with

$$U_0 = \int_e^f u_0(\in)\Delta\in > 0 \text{ if } f \in C^0(I,R) \text{ and } \overline{f} > 1,$$

$$h_{\overline{f}}^0(\Upsilon) \le \int_e^f v(\in)\log\frac{1}{u(\in)}\Delta(\in) + \log(u_0) \qquad (2.6)$$

Where $h_{\overline{f}}^0(\Upsilon)$ is defined in the equation (6) and $e, f \in T^0$

Remark 2.2: when the base of log less than 1 then the inequality (2.6) holds in the opposite direction for.

Example 2.4: take $T^0 = h^0 z, h^0 > 0$ in the theorem (2.2) to get an upper bound for entropy in h^0 -discrete calculus.

$$\sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0 \log\left(\frac{u\left(l_0h^0\right)h^0+v\left(l_0h^0\right)h^0}{2v\left(l_0h^0\right)h^0}\right) \le \sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0 \log\left(\frac{u\left(l_0h^0\right)h^0+v\left(l_0h^0\right)h^0}{2v\left(l_0h^0\right)h^0}\right)+\left(\sum_{l_0=e/h^0}^{f/h^0-1} u\left(l_0h^0\right)h^0\right) \tag{2.7}$$

Example 2.5: take $T^0 = p^{N_0}\left(p>1\right)$ in the theorem (2.2) then we have

$$\sum_{l_0=0}^{n-1} p^{l_0+1} v\left(l_0\right)\log\left(\frac{u\left(l_0\right)+v\left(l_0\right)}{2v\left(l_0\right)}\right) \le \sum_{l_0=0}^{n-1} p^{l_0+1} v\left(l_0\right)\log\left(\frac{u\left(l_0\right)+v\left(l_0\right)}{2v\left(l_0\right)}\right)+\log\left(p^{l_0+1}u\left(p^{l_0}\right)\right) \tag{2.8}$$

Remark 2.3: above equation contains Shannon entropy.

iii. The symmetric chi-square divergence measure

If $f\left(j\right)=\dfrac{j\left(j-1\right)^2}{\left(2j-1\right)}\ \forall j > 1/2$, then the divergence measure is given by

$$S_f\left(U,V\right)=\frac{1}{8}\left[\sum_{i=1}^{v}\frac{\left(u_i+v_i\right)\left(u_i-v_i\right)^2}{u_iv_i}\right]=\frac{1}{8}\Psi_{\chi^2}\left(U,V\right)$$

The χ^2- divergence measure is defined on the time scale as follows

$$\frac{1}{8}\Psi_{\chi^2}\left(U,V\right)=\frac{1}{8}\int_e^f\left[\frac{\left(u\left(\in\right)+v\left(\in\right)\right)\left(u\left(\in\right)-v\left(\in\right)\right)^2}{u\left(\in\right)v\left(\in\right)}\right]\Delta\in \tag{2.9}$$

Theorem 2.3: Using the theorem (2.1) then

$$\frac{1}{U}\left[U^2-V^2\right]\le\psi_{\chi^2}\left(U,V\right) \tag{2.10}$$

Where $\psi_{\chi^2}\left(U,V\right)$ is defined in the equation (2.8)

Proof: now suppose $f\left(\in\right)=\in^2-1$,

$$\left(\frac{U}{V}\right)^2-1\le\frac{1}{V}\int_e^f\left[\frac{\left(u\left(\in\right)+v\left(\in\right)\right)\left(u\left(\in\right)-v\left(\in\right)\right)^2}{u\left(\in\right)v\left(\in\right)}\right]\Delta\in$$

Simplify the above equation, then we get

$$U^2-V^2\le V\int_e^f\left[\frac{\left(u\left(\in\right)+v\left(\in\right)\right)\left(u\left(\in\right)-v\left(\in\right)\right)^2}{u\left(\in\right)v\left(\in\right)}\right]\Delta\in$$

This is the required result

Example 2.6: if $T^0=R$ (set of real number), then the equation (2.9) is

$$\frac{1}{\int_e^f v\left(\in\right)d\in}\left[\left(\int_e^f u\left(\in\right)d\in\right)^2-\left(\int_e^f v\left(\in\right)d\in\right)^2\right]\le\int_e^f\frac{\left(u\left(\in\right)+v\left(\in\right)\right)\left(u\left(\in\right)-v\left(\in\right)\right)^2}{u\left(\in\right)v\left(\in\right)}\Delta\in$$

Example 2.7: Putting $T^0=h^0Z, h^0>0$ in theorem (2.3) then

$$\frac{1}{\sum_{l_0=e/h^0}^{f/h_0-1} v(l_0h^0)h^0}\left[\left(\sum_{l_0=e/h^0}^{f/h_0-1} u(l_0h^0)h^0\right)^2-\left(\sum_{l_0=e/h^0}^{f/h_0-1} v(l_0h^0)h^0\right)^2\right]\le\sum_{l_0=e/h^0}^{f/h_0-1}\frac{\left(u(l_0h^0)h^0+v(l_0h^0)\right)\left(u(l_0h^0)h^0-v(l_0h^0)h^0\right)^2}{u(l_0h^0)v(l_0h^0)}$$

(2.11)

Remark 2.4: now take $h^0=1$ in the above equation, let $e=0,\ f=n, v(i)=v_j$ and $u(i)=u_j$ to get χ^2- divergence

$$\frac{1}{\sum_{j=1}^{n} v_j}\left[\left(\sum_{j=1}^{n} u_j\right)^2-\left(\sum_{j=1}^{n} v_j\right)^2\right]\le\psi_{\chi^2}(U,V)$$

Where $$\psi_{\chi^2}(U,V)=\sum_{j=1}^{n}\left[\frac{(u_j+v_j)(u_j-v_j)}{u_jv_j}\right] \qquad (2.12)$$

Remark 2.5: now take $T^0=p^{N_0}\ (p>1)$ for χ^2- divergence

$$\frac{1}{\sum_{l_0=0}^{n-1} p^{l_0+1}v(p^{l_0})}\left[\left(\sum_{l_0=0}^{n-1} p^{l_0+1}u(p^{l_0})\right)^2-\left(\sum_{l_0=0}^{n-1} p^{l_0+1}v(p^{l_0})\right)^2\right]\le\sum_{l_0=0}^{n-1} p^{l_0+1}\frac{\left(u(p^{l_0})+v(p^{l_0})\right)\left(u(p^{l_0})-v(p^{l_0})\right)^2}{u(p^{l_0})v(p^{l_0})}$$

(2.13)

iv. Jensen-Shannon divergence measure

$f(j)=-\log(j)$, this is known as Jensen-Shannon divergence measure

$$S_f(U,V)=\sum_{i=1}^{n} v_i\log\left(\frac{2v_i}{u_i+v_i}\right)=F(U,V)$$

Jensen-Shannon divergence on time scale is characterized as.

$$F(U,V)=\int_e^f v(\in)\log\left(\frac{2u(\in)}{u(\in)+v(\in)}\right)\Delta\in \qquad (2.14)$$

Theorem 2.4: apply the conditions in theorem (2.1),

$$U\ln\left(\frac{U}{V}\right)\le F(U,V)$$

(2.15)

Where $F(U,V)$ defined as follows in this equation

Proof: let $f(\in)=\in\log(\in)$ in the equation (2.2) now get

$$\frac{U}{V}\ln\left(\frac{U}{V}\right)\le\frac{1}{u}\int_e^f v(\in)\ln\left(\frac{2u(\in)}{u(\in)+v(\in)}\right)\Delta\in$$

and

$$U\ln\left(\frac{U}{V}\right)\le\int_e^f v(\in)\ln\left(\frac{2u(\in)}{u(\in)+v(\in)}\right)\Delta\in$$

This is the required result

Example 2.8: for $T^0=R$ (set of real number) then the equation (2.14) becomes

$$\int_e^f v(\in)d\in \ln\left(\frac{2\int_e^f u(\in)d\in}{\int_e^f u(\in)d\in+\int_e^f v(\in)d\in}\right) \le \int_e^f v(\in)\ln\left(\frac{2v(\in)}{u(\in)+v(\in)}\right)\Delta\in$$

Example 2.9: take $T^0 = h^0 Z, h^0 > 0$ in theorem (2.4) to get a lower bounds in h^0 – discrete calculus.

$$\sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0 \ln\left(\frac{2\sum_{l_0=e/h^0}^{f/h^0-1} u\left(l_0h^0\right)h^0}{\sum_{l_0=e/h^0}^{f/h^0-1} u\left(l_0h^0\right)h^0+\sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0}\right) \le \sum_{l_0=e/h^0}^{f/h^0-1} v\left(l_0h^0\right)h^0 \ln\left(\frac{2u\left(l_0h^0\right)h^0}{u\left(l_0h^0\right)h^0+v\left(l_0h^0\right)h^0}\right) \tag{2.16}$$

Remark 2.6: now take $h^0 = 1$ in the above equation, let $e = 0, f = n, v(i) = v_j$ and $u(i) = u_j$ to get Jensen-Shannon divergence.

$$\sum_{j=1}^{n} v_j \ln\left(\frac{2\sum_{j=1}^{n} v_j}{\sum_{j=1}^{n} u_j+\sum_{j=1}^{n} v_j}\right) \le F(U,V)$$

Where

$$F(U,V) = \sum_{j=1}^{n} v_j \ln\left(\frac{2v_j}{u_j+v_j}\right) \tag{2.17}$$

Remark 2.7: For the lower bound of quantum calculus, putting $T^0 = p^{N_0}\ (p>1)$ in the theorem (2.4) to get

$$\sum_{l_0=0}^{n-1} p^{l_0+1}v\left(p^{l_0}\right)\ln\left(\frac{2\sum_{l_0=0}^{n-1} p^{l_0+1}v\left(p^{l_0}\right)}{\sum_{l_0=0}^{n-1} p^{l_0+1}u\left(p^{l_0}\right)+\sum_{l_0=0}^{n-1} p^{l_0+1}v\left(p^{l_0}\right)}\right) \le \sum_{l_0=0}^{n-1} p^{l_0+1}v\left(p^{l_0}\right)\log\left(\frac{2v\left(p^{l_0}\right)}{u\left(p^{l_0}\right)+v\left(p^{l_0}\right)}\right)$$

v. Hellinger discrimination

If $f(j) = \left(1-\sqrt{j}\right)$, this is known as Hellinger discrimination

$$S_f(U,V) = h\left(\frac{U+V}{2}, U\right)$$

Hellinger discrimination on time scale

$$H^2(U,V) = \int_e^f v(\in)\left(\sqrt{2v(\in)}-\sqrt{u(\in)+v(\in)}\right)^2 \Delta\in \tag{2.18}$$

Theorem2.5: apply the conditions of theorem (2.1) to obtain

$$v\left(\sqrt{2v}-\sqrt{u+v}\right)^2 \le H^2(u,v) \tag{2.19}$$

where $H^2(u,v)$ is defined as above.

Proof: let $f(\in) = \frac{1}{2}\left(\sqrt{\in}-1\right)^2$ in the equation (2.2) to get

$$\frac{1}{2}\left(\sqrt{2}-\sqrt{\frac{u}{v}+1}\right)^2 \le \frac{1}{2}\int_e^f v(\in)\left(\sqrt{\frac{u(\in)+v(\in)}{2v(\in)}}-1\right)^2 \Delta\in \tag{2.20}$$

After simplification we obtain

$$\left(\sqrt{2v}-\sqrt{u+v}\right)^2 \le \int_e^f \left(\sqrt{2v(\in)}-\sqrt{u(\in)+v(\in)}\right)^2 \Delta\in$$

This is the required result.

Example 2.10: for $T^0 = R$ (set of real number) the equation (2.19) becomes.

$$\frac{1}{2}\left[\left(\int_e^f 2v(\in)d\in\right)^{1/2} - \left(\int_e^f \left(v(\in)+u(\in)\right)d\in\right)^{1/2}\right]^2 \le \frac{1}{2}\int_e^f \left(\sqrt{2v(\in)}-\sqrt{v(\in)+u(\in)}\right)^2 d\in$$

Example 2.11: take $T^0 = h^0 Z, h^0 > 0$ in (2.4)

$$\frac{1}{2}\left[\left(2\sum_{l_0=e/h_0}^{f/h_0-1} v(l_0 h_0)h_0\right)^{1/2} - \left(\sum_{l_0=e/h_0}^{f/h_0-1} u(l_0 h_0)h_0 + \sum_{l_0=e/h_0}^{f/h_0-1} v(l_0 h_0)h_0\right)^{1/2}\right]^2 \le \frac{1}{2}\sum_{l_0=e/h_0}^{f/h_0-1}\left(\sqrt{2v(l_0 h_0)h_0}-\sqrt{u(l_0 h_0)h_0+v(l_0 h_0)h_0}\right)^2 \tag{2.21}$$

Remark 2.9: now take $h^0 = 1$ in the above equation, let $e = 0, f = n, v(i) = v_j$ and $u(i) = u_j$ to get Hellinger - discriminate.

$$\frac{1}{2}\left[\left(\sum_{j=1}^n 2v_j\right)^{1/2} - \left(\sum_{j=1}^n u_j + \sum_{j=1}^n v_j\right)^{1/2}\right]^2 \le H^2(U,V) \tag{2.22}$$

Remark 2.10: For the lower bound of Hellinger discrimination, putting $T^0 = p^{N_0}\ (p>1)$ and using theorem (2.5) to get a lower bound of.

$$\frac{1}{2}\left[\left(2\sum_{l_0=0}^{n-1} p^{l_0+1}v(p^{l_0})\right)^{1/2} - \left(\sum_{l_0=0}^{n-1} p^{l_0+1}u(p^{l_0}) + \sum_{l_0=0}^{n-1} p^{l_0+1}v(p^{l_0})\right)^{1/2}\right]^2 \le \frac{1}{2}\sum_{l_0=0}^{n-1} p^{l_0+1}\left(\sqrt{2v(p^{l_0})}-\sqrt{u(p^{l_0})+v(p^{l_0})}\right)^2 \tag{2.23}$$

vi. Variational distance

$f(j) = |j-1|$ for $j > 0$ this is known as Variational distance

$$S_f(U,V) = \frac{1}{2}\sum_{i=1}^n |u_i - v_i| = \frac{1}{2}V(u,v)$$

and

$$\frac{1}{2}V(u,v) = \frac{1}{2}\int_e^f |u(\in)-v(\in)|\Delta\in \tag{2.24}$$

Theorem 2.6: apply the conditions of theorem (2.1) to get

$$\frac{1}{2}|u(\in)-v(\in)| \le V(u,v)$$

where $V(u,v)$ is defined above

Proof : let $f(\in) = |\in -1|$ in the equation (2.2) to get

$$\frac{1}{2}v\left(\frac{u}{v}-1\right) \le \frac{1}{2}\int_e^f v(\in)\left(\frac{u(\in)}{v(\in)}-1\right)\Delta\in$$

and

$$\frac{1}{2}\left|u(\in)-v(\in)\right| \le \int_{e}^{f}\left|u(\in)-v(\in)\right|\Delta\in$$

This is the required result

Example 2.12: for $T^0 = R$ (set of real number) takes the form

$$\frac{1}{2}\left|\int_{e}^{f}u(\in)d\in-\int_{e}^{f}v(\in)d\in\right| \le \int_{e}^{f}\left|u(\in)-v(\in)\right|\Delta\in$$

Example 2.13: For the lower bound of variational distance, $T^0 = h^0 Z, h^0 > 0$ in theorem (2.5)

$$\frac{1}{2}\left|\sum_{l_0=e/h_0}^{f/h_0 -1} u(l_0 h_0)h_0 - \sum_{l_0=e/h_0}^{f/h_0 -1} v(l_0 h_0)h_0\right| \le \sum_{l_0=e/h_0}^{f/h_0 -1}\left|u(l_0 h_0)h_0 - v(l_0 h_0)h_0\right| \tag{2.25}$$

Remark 2.11: now take $h^0 = 1$ in the above equation, let $e=0, f=n, v(i)=v_j$ and $u(i)=u_j$ to get Variational distance.

$$\frac{1}{2}\left(\left|\sum_{j=1}^{n}u(\in)-\sum_{j=1}^{n}v(\in)\right|\right) \le V(U,V)$$

Remark 2.12: now take $T^0 = p^{N_0}\ (p>1)$ in the theorem (2.6) to get a lower bound for Variational-distance.

$$\frac{1}{2}\left(\left|\sum_{l_0=0}^{n-1}p^{l_0+1}u\left(p^{l_0}\right)-\sum_{l_0=0}^{n-1}p^{l_0+1}u\left(p^{l_0}\right)\right|\right) \le \sum_{l_0=0}^{n-1}p^{l_0+1}\left|u\left(p^{l_0}\right)-v\left(p^{l_0}\right)\right| \tag{2.26}$$

vii. Triangular discrimination :

if $f(j)=\frac{(j-1)^2}{j}$ for $j>0$, triangular discrimination known as follows

$$S_f(U,V)=\frac{1}{2}\sum_{i=1}^{n}\frac{(u_i-v_i)^2}{u_1+v_i}=\frac{1}{2}\Delta(U,V)$$

Triangular discrimination on the scale is characterized as

$$\Delta(U,V)=\frac{1}{2}\int_{e}^{f}\frac{\left[u(\in)-v(\in)\right]^2}{u(\in)+v(\in)}\Delta\in \tag{2.27}$$

Theorem 2.7:

Apply the conditions for obtain $\frac{1}{2}\frac{[U-V]^2}{[U+V]} \le \Delta(u,v)$, using theorem (2.5), where $\Delta(u,v)$ defined as above.

Proof: $f(\in)=\frac{(\in-1)^2}{\in}$ in the equation (2.2) to get

$$V\left(\frac{\left(\frac{U+V}{2V}-1\right)^2}{\frac{U+V}{2V}}\right) \le \int_{e}^{f}v(\in)\frac{\left(\frac{u(\in)+v(\in)}{2v(\in)}-1\right)^2}{\frac{u(\in)+v(\in)}{2v(\in)}}d\in$$

Or

$$\frac{(U-V)^2}{(U+V)} \leq \int_e^f \frac{\left[u(\in)-v(\in)\right]^2}{u(\in)-v(\in)} \Delta \in \qquad (2.28)$$

Example 2.14: for $T^0 = R$ (set of real number) takes the form

$$\frac{\left[\int_e^f u(\in) d\in - \int_e^f v(\in) d\in\right]^2}{\int_e^f u(\in) d\in + \int_e^f v(\in) d\in} \leq \int_e^f \frac{\left[u(\in)-v(\in)\right]^2}{u(\in)+v(\in)} d\in$$

Example 2.15: For the lower bound of Triangular discrimination, $T^0 = h^0 Z, h^0 > 0$ in theorem (2.5)

$$\frac{\left(\sum_{l_0=e/h_0}^{f/h_0-1} u(l_0h_0)h_0 - \sum_{l_0=e/h_0}^{f/h_0-1} v(l_0h_0)h_0\right)^2}{\sum_{l_0=e/h_0}^{f/h_0-1} u(l_0h_0)h_0 + \sum_{l_0=e/h_0}^{f/h_0-1} u(l_0h_0)h_0} \leq \sum_{l_0=e/h_0}^{f/h_0-1} \frac{h_0\left(u(l_0h_0)-v(l_0h_0)\right)^2}{u(l_0h_0)+v(l_0h_0)}$$

(2.29)

Remark 2.13: now take $h^0 = 1$ in the above equation let $e = 0, f = n, v(l_0) = v_j$ and $u(l_0) = u_j$ to get Triangular discrimination.

$$\frac{\left(\sum_{j=1}^{n} u_j - \sum_{j=1}^{n} v_j\right)^2}{\sum_{j=1}^{n} u_j + \sum_{j=1}^{n} u_j} \leq \Delta(U,V)$$

Where

$$\Delta(U,V) = \sum_{j=1}^{n} \frac{\left(u_j - v_j\right)^2}{u_j + v_j}$$

Remark 2.14: now take $T^0 = p^{N_0}\left(p > 1\right)$ in the theorem (2.6) to get a lower bound for Tringular discrimination.

$$\frac{\left(\sum_{l_0=0}^{n-1} p^{l_0+1} u\left(p^{l_0}\right) - \sum_{l_0=0}^{n-1} p^{l_0+1} v\left(p^{l_0}\right)\right)^2}{\sum_{l_0=0}^{n-1} p^{l_0+1} u\left(p^{l_0}\right) + \sum_{l_0=0}^{n-1} p^{l_0+1} v\left(p^{l_0}\right)} \leq \sum_{l_0=0}^{n-1} p^{l_0+1} \frac{\left[u\left(p^{l_0}\right)-v\left(p^{l_0}\right)\right]^2}{u\left(p^{l_0}\right)+v\left(p^{l_0}\right)} \qquad (2.30)$$

3. Conclusion

Established new f-divergence on the time scale and also discussed the particular cases. The particular cases may be useful in the literature of information theory, inequalities and special function. The integral inequalities and discrete inequalities can be used the attractive results of information theory.

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