Improved Oscillation and Asymptotic Conditions forThird-Order Nonlinear Neutral Type Difference Equations

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Abstract. For a class of advancedthird-order neutral typenonlineardifference equations, we improved the notion of oscillation and asymptotic criterion. The findings are unique, and they improve and complement previous findings in the literature. We propose some new criteria for ensuring that every solution is oscillatory by applying extended Ricatti type transformation. The importance of the primary results is demonstrated by certain instances.

INTRODUCTION

Here we investigate the oscillatory and asymptotic behaviorconditionsfor third-order nonlinear difference equation.

$$\Delta\left(\alpha(\eta)\left(\Delta^2(\gamma(\eta)+\beta(\eta)\gamma(\eta-\sigma))\right)^\rho\right)+\delta(\eta)\gamma^\rho(\eta-\mu)=0, \eta\geq\eta_0 \tag{1.1}$$

where$\{\alpha(\eta)\}$ is a positive real sequence with $\sum_{\eta=\eta_0}^{\infty}\frac{1}{\alpha(\eta)^{\frac{1}{\rho}}}<\infty$for all $\eta\geq\eta_0$,$\{\delta(\eta)\}$is a nonnegative real sequence and $\{\gamma(\eta)\}$ is a bounded nonnegative real sequence.

$(\hbar 1)\{\alpha(\mu)\}_{\mu=\mu_0}^{\infty}and\{\gamma(\mu)\}_{\mu=\mu_0}^{\infty}$are sequences of real numbers

$$\sum_{\eta=\eta_0}^{\infty}\left(\frac{1}{\alpha(\eta)}\right)=\sum_{\eta=\eta_0}^{\infty}\left(\frac{1}{\gamma(\eta)}\right)=\infty, \tag{1.2}$$

$(\hbar 2) 0\leq\beta(\eta)<1, \delta(\eta)\geq 0$and $\delta(\mu)\}_{\eta=\eta_0}^{\infty}$has a positive subsequence

$(h3)\gamma^\rho: X\to X$is continuous function such that $\gamma^\rho(\eta-\mu)\geq R>0$

$(h4)\rho$is a ratio of odd positive integers and σ and μ are nonnegative integers.

Let $\varphi = max(\sigma,\mu)$, be the maximum value. A real classification$\{\gamma(\eta)\}$ defined for every $\eta\geq 1-\varphi$in (1.1) for all $\eta\in N$. If a nontrivial solution of (1.1) is neither gradually positive nor finally negative, it is said to be oscillatory; otherwise, it is said to be nonoscillatory. If all of the conditions of (1.1) are oscillatory or approach to zero as$\eta\to\infty$, it is said to be virtually oscillatory.

The majority of findings for the oscillation and asymptoticconditionsof third order nonlinearneutraltypedifference equations are derived under the assumption $-1<\beta(\eta)<1$, according to a survey of the literature. As a result, it's fascinating to investigate the oscillatory behaviour (1.1) under the condition $0\leq\beta(\eta)\leq\beta<\infty$. The equation (1.1) must meet certain characteristics in order to be considered virtually oscillatory. A lot of interest exposed in the asymptotic and oscillatory behaviour of solutions to nonlinear neutral type difference equations during the last three decades; see, for example, [1, 2, 8, 10, 11, 13, 15] and the publications mentioned therein for recent results of this sort. However, only a few results on the oscillation of advanced type difference equations have been published (see [3, 5–7, 9, 12]).According to a review of the literature, all of the results established in [14, 10, 12, 13] for the difference equations of neutral type guarantees that every solution is eitheroscillatory or tends to zero monotonically. As a consequence, the results obtained in this study are superior to those reported in [10, 12, 14, 15]. The following is a breakdown of how this article is structured. The major results are deduced in Section 2, and some examples are provided in Section 3 to demonstrate the significance of the primary conclusions.

PRELIMINARIES

In this section, we present the main improved and sufficient conditionsof oscillation and asymptotic conditions for (1.1) by Riccati transformation approach, which ensures that every solution $\{\gamma(\eta)\}$ of (1.1) oscillates.

Lemma 2.1. Let $\{\gamma(\eta)\}$be aeventually positive solution of (1.1). Then there exist two cases for $\{z(\eta)\}$
(i) $z(\eta) > 0,\ \Delta z(\eta) > 0, \Delta^2 z(\eta) > 0, \Delta(\alpha(\eta)(\Delta^2 z(\eta)^\alpha) \le 0$;
(ii)$z(\eta) > 0,\ \Delta z(\eta) < 0, \Delta^2 z(\eta) > 0, \Delta(\alpha(\eta)(\Delta^2 z(\eta)^\alpha) \le 0$;
The proof of the lemma is trivial.

MAIN RESULTS

Theorem 2.1. Assume that (h1) − (h4) holds. Moreover if there exists a positive sequence $\{\beta(\eta)\}_{\eta=\eta_0}^{\infty}$ such that,

$$\lim_{\eta\to\infty}\sup\sum_{s=\eta_0}^{\eta}\left[C\vartheta(s)y(s)(1-x(s-\mu)^\rho-\frac{(\Delta\vartheta(s))^2}{2^{3-\rho}\psi(s-\mu)\vartheta(s)}\right]=\infty \tag{2.1}$$

Then every solution of $\{\gamma(\eta)\}$in (1.1)satisfiesoscillatory and asymptotic conditions.

Proof: Let $\{\gamma(\eta)\}$is a non-oscillatory solution of (1.1). We can assume that $\gamma(\eta-\mu) > 0$ for$\eta \ge \eta_1$ where $\eta_1$is selected without losing generality. We will just analyse this instance because the evidence when $\gamma(\eta) < 0$is equivalent. If $z(\eta)$is defined as in (2.1), then $z(\eta) > 0$and there are two possible cases based on Lemma 2.1. Consider Case (i) $z(\eta)$is a positive solution to the neutral type difference inequality. By using the Riccati substitution, define the sequence $\{\omega(\eta)\}$.

$$\omega(\eta)=\vartheta(\eta)\frac{\alpha(\eta)\Delta\big(\Delta z(\eta)\big)^\rho}{z^\rho(\eta-\mu)}, \eta\ge\eta_1 \tag{2.2}$$

When $\omega(\eta) > 0$ and

$$\Delta\omega(\eta)=\alpha(\eta+1)\Delta\big(\Delta z(\eta+1)\big)^\rho\Delta\left[\frac{\vartheta(\eta)}{z^\rho(\eta-\mu)}\right]+\frac{\vartheta(\eta)\Delta\Big(\alpha(\eta)\Delta\big(\Delta z(\eta)\big)^\rho\Big)}{z^\rho(\eta-\mu)}$$

which implies

$$\Delta\omega(\eta)\le-\vartheta(\eta)X(\eta)+\frac{\Delta\vartheta(\eta)}{\vartheta(\eta+1)}\omega(\eta+1)$$
$$-\frac{\vartheta(\eta)\alpha(\eta+1)\Delta\big(\Delta z(\eta+1)\big)^\rho\Delta z^\rho(\eta-\mu)}{z^\rho(\eta-\mu)z^\rho(\eta-\mu+1)} \tag{2.3}$$

where $X(\eta)=Cx(\eta)\big(1-y(\eta-\mu)\big)^\rho$. From Lemma 2.1 case(i), we have $z(\eta-\mu+1)\ge z(\eta-\mu)$. Then from (2.3), we obtain

$$\Delta\omega(\eta)\le-\vartheta(\eta)X(\eta)+\frac{\Delta\vartheta(\eta)}{\vartheta(\eta+1)}\omega(\eta+1)$$
$$-\frac{\vartheta(\eta)\alpha(\eta+1)\Delta\big(\Delta z(\eta+1)\big)^\rho\Delta(z^\rho(\eta-\mu))}{z^{2\rho}(\eta-\mu+1)} \tag{2.4}$$

Using inequality
$a^{\lambda} - b^{\lambda} \geq 2^{1-\lambda}(a-b)^{\lambda}$for all $a \geq b > 0$ and $\lambda \geq 1$,
We have

$$\Delta\big(z^{\rho}(\eta-\mu)\big) = z^{\rho}(\eta-\mu+1) - z^{\rho}(\eta-\mu) \geq 2^{1-\lambda}(z(\eta-\mu+1) - z(\eta-\mu))^{\rho}$$
$$= 2^{1-\lambda}\big(\Delta z(\eta+1)\big)^{\rho}, \rho \geq 1. \quad (2.5)$$

$$\Delta\omega(\eta) \leq -\vartheta(\eta)X(\eta) + \frac{\Delta\vartheta(\eta)}{\vartheta(\eta+1)}\omega(\eta+1) - 2^{1-\rho}\frac{\vartheta(\eta)\alpha(\eta+1)\Delta\big(\Delta z(\eta+1)\big)^{\rho}\big(\Delta z(\eta-\mu)\big)^{\rho}}{z^{2\rho}(\eta-\mu+1)} \quad (2.6)$$

Where $\psi(\eta-\mu) = \Delta z^{\rho}(\eta-\mu)$

$$\Delta\omega(\eta) \leq -\vartheta(\eta)X(\eta) + \frac{\Delta\vartheta(\eta)}{\vartheta(\eta+1)}\omega(\eta+1) - 2^{1-\rho}\frac{\vartheta(\eta)\alpha(\eta+1)\Delta\big(\Delta z(\eta+1)\big)^{\rho}\psi(\eta-\mu)}{z^{2\rho}(\eta-\mu+1)} \quad (2.7)$$

$$\Delta\omega(\eta) \leq -\vartheta(\eta)X(\eta) + \frac{\Delta\vartheta(\eta)}{\vartheta(\eta+1)}\omega(\eta+1) - 2^{1-\rho}\frac{\vartheta(\eta)\psi(\eta-\mu)}{\vartheta^2(\eta+1)}\omega^2(\eta+1) \quad (2.8)$$

By completing the square, we get

$$\Delta\omega(\eta) < -\left[\vartheta(\eta)X(\eta) - \frac{\big(\Delta\vartheta(\eta)\big)^2}{2^{3-\rho}\psi(\eta-\mu)\vartheta(\eta)}\right] \quad (2.9)$$

Summing (2.9) from $\eta_1$to η, we obtain

$$-\omega(\eta_1) < \omega(\eta+1) - \omega(\eta_1) < -\sum_{s=\eta_1}^{\eta}\left[\vartheta(s)y(s) - \frac{(\Delta\vartheta(s))^2}{2^{3-\rho}\psi(s-\mu)\vartheta(s)}\right]$$

$$\sum_{s=\eta_1}^{\eta}\left[\vartheta(s)y(s) - \frac{(\Delta\vartheta(s))^2}{2^{3-\rho}\psi(s-\mu)\vartheta(s)}\right] < c_1 \quad (2.10)$$

for all large η, and this is contrary to (2.1). If the Case (ii) holds.Hence the proof.

Conclusion

This paper deals the improved oscillation and asymptotic conditions for 3rd order Nonlinear Neutral Type Difference Equations with deviation for parameter estimation under conditions.

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