HOMOTOPICALLY DENSE PROPERTIES OF THE ALEXANDROV COMPACTIFICATION OF SOME SUBSPACES OF THE SPACE OF PROBABILITY MEASURES

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Abstract
In this paper, we consider topological and extensor properties of the one-point and Alexandrov compactification of αX for a locally compact space X, in the case of subspaces of the space $P(X)$ of all probability measures. Getting the following results:

1) $\alpha : LComp \to Comp$ is a covariant functor;

2) For any locally compact ANR space X its compactification αX takes place:

a) X is homotopy dense in αX;
b) the embedding of X in αX is a fine homotopy equivalence;
c) the set $C(Q, X)$ is everywhere dense in $C(X, \alpha X)$;

3) For any locally compact X space, X is ANR if and only if the compactification of αX is an ANR space.

4) The compactification αM of any $Q-$manifold M is the Hilbert cube Q i.e. $\alpha M;\ Q$.

5) For any infinite compact set X;

a) the compactification $\alpha(P(X) \setminus P(A))$ of the space $P(X) \setminus P(A)$ is homeomorphic to Q, where $A \subset X, A \neq X, \overline{A} = A$; those. $\alpha(P(X) \setminus P(A));\ Q$;

b) $\alpha(S_P(A))$ is homeomorphic to Q, where $A \subset X, A \neq X$ A is open in X; i.e. $\alpha(S_P(A));$ Q;
c) $\alpha(P_{\nabla n}(X));$ Q, where $n \in N$ and $P_{\nabla n}(X) = P(X) \setminus P_n(X)$;
d) $\alpha(P_{f,n}(X));$ Q, where $n \in N$ and $P_{\nabla f,n}(X) = P(X) \setminus P_{f,n}(X)$;
e) $\alpha(P(X) \setminus P_f(X));$ Q.

6) For any infinite compact set X, the compactification $\alpha(P_\omega(X))$ is homeomorphic to the Hilbert cube Q. those. $\alpha(P_\omega(X));$ Q.

7) Let X be an infinite compact $A_1, A_2, ..., A_n, ...$ closed subsets of X such that $\bigcup_{i=1}^{\infty} A_i = A$ is everywhere dense in X and A is not open in X. Then the $\bigcup_{i=1}^{\infty} S_P(A_i)$ of the manifold contains the homeomorphic manifold $\sigma \times S$.

8) Let X be a connected locally compact locally connected space, then X is homotopy dense in αX and the identity embedding $i_\alpha : X \to \alpha X$ is a homotopy equivalence.

9) For any infinite compact set X, the one-point extension $\alpha P_\omega(X)$ of the space $P_\omega(X)$ is homeomorphic to the Hilbert cube Q.

10) For any locally compact infinite space X, the space $\alpha P_\omega(\alpha X)$ is homeomorphic to the Hilbert cube Q.

Key words. Compactification, $A(N)R$ -space, homotopy density, probability measures, manifolds, covariant functor, fine homotopy equivalence.

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Introduction

A space X is σ-compact if it can be represented as the union of at most a countable number of its compact subsets.

Definition [1]. A continuous mapping $f : X \to Y$ is said to be proper if the inverse image of every compact subset K of the space Y is compact.

Definition [1]. A closed mapping $f : X \to Y$ is called perfect if it is compact, i.e. the inverse images of all one-point subsets are compact.

Proposition 4.13[1]. Let $f : X \to Y$ be a proper mapping from X to a locally compact space Y, then X is also locally compact.

It is known that a Hausdorff space X is locally compact in this and only in this case if it is homeomorphic to an open subset of a compact set [1]. On the other hand, for a locally compact Hausdorff space X to be σ compact, it is necessary and sufficient that in the one-point compactification of αX, the point at infinity ω has a countable local base, i.e. countable fundamental system of open neighborhoods [1].

Theorem [1]. One point ω can be attached to any locally compact space X (and only such a space) in such a way that the compact $\alpha X = X \bigcup \{\omega\}$ is obtained, and the topology in X as in a subspace of the compact set αX, will coincide with the topology given in X, while the topology in αX is uniquely determined by the topology in X.

Note that neighborhoods of X are open in αX, and neighborhoods of ω form sets of the form $\omega \bigcup (X \setminus B)$, where B is a compact subset of X. These extensions of αX can be obtained by Tikhonov's method with the help of a splitter.

A family consisting of functions $f : X \to [0,1]$ c functions equal to zero in outside some compact set. This extension is αX -smallest. Since the space X is locally compact, it is open in any of its compact extensions in X. Hence, the outgrowth in $X \setminus X$ is closed. Consider a partition of the space X whose only non-one-point element is the set in $X \setminus X$. The quotient space of a space in X is obviously homeomorphic to αX with respect to the partition, and the corresponding quotient of – mapping is the natural mapping of an extension in X onto the Alexander extension αX.

Let the $Comp$-category of compact spaces and continuous mappings into themselves (objects are compact spaces, and morphisms are continuous mappings), and $LComp$-the category of locally compact spaces and continuous proper mappings into themselves (objects are locally compact spaces, morphisms - own mappings). It is clear that $Comp$ is a complete subcategory of the category $LComp$.

By virtue of Aleksandrov's theorem on the one-point (minimal) compactification $\alpha(X)$ of a locally compact space, the transition X to $\alpha(X)$ and from f to $\alpha(f)$ is a covariant functor from $LComp$ to $Comp$. those. $\alpha : LComp \to Comp$.

a) If $f : X \to Y$, then $\alpha(f) : \alpha(X) \to \alpha(Y)$;

b) $id_X : X \to X$, then $\alpha(id_X) = id_{\alpha(X)}$;

c) *If* $f : X \to Y$ and $g : Y \to Z$ and $h = g \circ f$ then $\alpha(h) = \alpha(g) \circ \alpha(f)$.

$$X \xrightarrow{f} Y \qquad \xrightarrow{\alpha} \qquad \alpha(X) \xrightarrow{\alpha(f)} \alpha(Y)$$

$$g \circ f \,]\quad [\; g \qquad\qquad \alpha(g) \circ \alpha(f) \,]\quad [\; \alpha(g)$$

$$Z \qquad\qquad\qquad\qquad \alpha(Z)$$

Recall that a topological space Y is called an absolute (neighborhood) retract in the class K (written) $Y \in A(N)R(K)$ [2] if $Y \in K$ and for every homeomorphism h mapping Y onto a closed subset $h(Y)$ of the space X from the class K, the set $h(Y)$ is a (neighbourhood) retract of the space X.

Recall that a topological space X is called a manifold modeled on the space Y, or a $Y-$manifold [2], if any point in the space X has a neighborhood homeomorphic to an open subset of the space X.

A Q-manifold is a separable metric space locally homeomorphic to the Hilbert cube Q, where $Q = \prod_{i=1}^{\infty} [-1,1]_i$ Hilbert cube [3], $W_i^{\pm} = \{(g_j) \in Q \mid g_i = \pm 1$ $j-$th face of the Hilbert cube Q, $BdQ = \bigcup_{i=1}^{\infty} W_i^{\pm} -$ is called the pseudoboundary of the Q cube, and $S = Q \setminus BdQ$ isthe pseudointerior of the Q cube.

Three objects play an important role in the theory of infinite-dimensional manifolds: the Hilbert cube Q, the separable Hilbert space ℓ_2 and $\sum -$ the linear span of the standard brick $Q' = \prod_{n=1}^{\infty} [0, \frac{1}{2^n}]$ in the Hilbert space ℓ_2, ℓ_2^f denotes a linear subspace of the Hilbert space space ℓ_2, consisting of all points, only a finite number of coordinates of which are different from zero, and Q^f is asubspace of the Hilbert cube Q, consisting of all points, only a finite number of coordinates of which are different from zero.

By the Anderson-Kadetz theorem [4], the Hilbert space ℓ_2 is homeomorphic to S [4]. It follows from Bessagi-Pelchinsky's results that $\sum$ is homeomorphic $rintQ$ [4]. Here $\mathrm{rint}Q$ stands for the set $\{x = (x_n) \in Q \mid |x_n| < t < 1$ for all $n \in N\}$. Further, it is obvious that $\mathrm{rint}Q \approx BdQ$ means that $BdQ \approx \sum$ is true.

It is known that the spaces $Q, \sum$ and ℓ_2 are strongly infinite-dimensional, while the spaces ℓ_2^f and Q_f are weakly infinite-dimensional, and all these spaces are uniform.

Definition [5]. A closed set A of space X is called Z -set in X if the identity mapping id_X of the space X can be arbitrarily closely approximated by the mappings $f: X \to X \setminus A$.

A countable union of Z -sets in X is called a $\sigma - Z -$set in X.

Following [6], a $\sigma - Z -$set, B of a Hilbert cube Q is called a boundary set in Q (denoted by $B(Q)$) if $Q \setminus B \approx \ell_2$. More generally, a boundary set in a Q-manifold is a $\sigma - Z -$set whose complement is a $\ell_2 -$manifold.

It follows from the above that the pseudo-boundary BdQ of the Hilbert cube Q is its boundary set.

Let X be a topological space.

A set $A \subset X$ is said to be homotopy dense in X [4] if there exists a homotopy $h(x,t): X \times [0,1] \to X$ such that $h(x,0) = id_X$ and $h(X \times (0,1]) \subset A$.

A a set $A \subset X$ is homotopy negligible in X if $X \setminus A$ is homotopy dense in X.

An embedding $e: Y \to X$ is homotopy dense (respectively, homotopy negligible) if $e(Y)$ is a homotopy dense set (respectively, homotopy negligible) in X [4].

For compact sets X there is a simple topological classification of the spaces $P(X)$ of all probability measures. In the case of a finite $n-$point space $X = \{n\}$, the points μ of the space $P(n) = P_n(n)$ are convex linear combinations of the Dirac measures:

$$\mu = m_0\delta(0) + m_1\delta(1) + ... + m_{n-1}\delta(n-1)$$

Therefore, they are naturally identified with points $(n-1)$ of the dimensional simplex σ^{n-1}. In this case, the Dirac $\delta(i)$ form the vertices of the simplex, and the masses m_i placed at the points i are the barycentric coordinates of the measure μ. Thus, the compact set $P(n)$ is affinely homeomorphic to the simplex σ^{n-1}.

In the case of an infinite compact set X, the space $P(X)$ is also a compact set [7]. Further, it, containing simplices of an arbitrarily large number of dimensions, is infinite-dimensional. By Cayley's theorem, a convex compact $P(X) \subset R^{C(X)}$ embeds affinely in ℓ_2. Therefore, by Keller's theorem, the compact $P(X)$, as an infinite-dimensional convex compact lying in ℓ_2, is homeomorphic to the Hilbert cube $Q = I^{\chi_0}$. On the other hand, the space $P(X)$ of all probability measures on X is called the set of all regular Borel probability measures on X equipped with the weakest topology for which every functional $f_u: C(X) \to R$ that maps the measure μ to $\mu(U)$ (U is an open set in X).

Let τ bean infinite cardinal number. A directed set is called $\tau -$complete if any chain of its elements, containing at most $\tau -$members, has a least upper bound in A.

A continuous spectrum defined over $\tau -$complete directed set is called $\tau -$complete [8].

For χ_0 spectrum and χ_0 completeness the names spectrum sigma and completeness sigma are used.

Definition [8]. A compact set homeomorphic to the limit space of some sigma-spectrum with open projections is said to be openly generated.

It is also known that any χ_1 -degrees of a non-one-point compact set K the space of all probability measures $P(K^{\chi_1})$ is homeomorphic to the Tikhonov cube I^{χ_1}, $P(K^{\chi_1}) = I^{\chi_1}$, $I-$ segment $[0,1]$. Note, in particular, that all these spaces are topologically homogeneous. A for spaces $P(K^{\chi_1})$ for $\tau > \chi_1$ the situation is different [8].

For an arbitrary compact set X and a measure $\mu \in P(X)$ its support $\text{supp}(\mu)$ is defined, which is the smallest of the closed sets $F \subset X$ for which $\mu(F) = \mu(X)$ i.e. $\text{supp}(\mu) = \bigcap\{A : A = \bar{A},\ \mu \in P(A)\}$;

$P_n(X) = \{\mu \in P(X) : |\text{supp}\,\mu| \le n\} -$ the set of all measures μ at most n supported, $P_\omega(X) = \bigcup_{n=1}^{\infty} P_n(X) -$ the set of all probability measures μ with finite supports. Recall that the space $P_f(X) \subset P(X)$ consists of all probability measures [9]

$$\mu = m_1\delta(x_1) + m_2\delta(x_2) + ... + m_n\delta(x_n)$$

with finite supports, for each of which $m_i \ge \dfrac{n}{n+1}$ for some i;

$$P_{f,n}(X) = \{\mu \in P_f(X) : |supp\mu| \le n\}.$$

Hence, the compact subspace $P_f(X)$ is also the union of the compact sets $P_{f,n}(X)$. those. $P_f(X) = \bigcup_{n=1}^{\infty} P_{f,n}(X)$. Obviously, $P_{f,n}(X) \subseteq P_n(X)$ and $P_f(X) \subseteq P_\omega(X)$.

Obviously, for a metric compact set X and any $n \in N$ the sets $P_n(X)$, $P_f(X)$, $P_{f,n}(X)$ are closed in $P(X)$.

Therefore, the subspaces $P_\omega(X) \subset P(X)$ and $P_\omega(X)$ are $\sigma-$ compact, everywhere dense in $P_\omega(X)$.

It is known that if X is an arbitrary Hausdorff locally compact but not compact space, then its Alexandrov compactification $\alpha(X)$, obviously serves as a compact Hausdorff extension of X with a one-point outgrowth $\{\omega\}$ the difference $\alpha(X) \setminus X$ is called an outgrowth. In particular, (S^n, f), where f is a stereographic projection, is a one-point compact Hausdorff extension of R^n, where S^n is an $n-$dimensional sphere in R^{n+1} [1].

Note that the unit ball B^n of the space R^n obviously serves as a compact extension of R^n, whose outgrowth is the sphere S^{n-1} and these extensions are obviously not equivalent to the extension (S^n, f) from the previous example, since the ball and sphere are not homeomorphic [1].

In particular, RP^1 (the projective line), just like the neighborhood S^1 , is a one-point Hausdorff compact extension of R^1 , and these extensions are equivalent due to the well-known Aleksandrov theorem.

However, for $n \ge 2$ the space RP^n and S^n, although also serve as Hausdorff extensions of R^n , but they are no longer equivalent (because they are not even homeomorphic to each other).

We know that one of the typical problems that led to the creation of the theory of compact extensions was the problem of the continuous continuation of a particular function given on a non-compact set. The point is that if a function f is given and is continuous on a non-compact subset X of a compact set H, then it does not always allow continuous extension, just its closure $\bar{X}$ in H and therefore there is a need for such a compact Hausdorff extension $b(X)$, to which one could continuously extend the original function [1].

We say that a subset U of a locally compact space X is bounded if its closure is $\overline{U}$ -compact [10].

It is known that the questions of extending a continuous mapping $f: X \to Y$ to some extension are closely related to *A(N)R* properties of topological spaces.

A classic example of a continuous function that cannot be extended to close its domain is the following function:

Let $H = [0,1] \subset R^1, X = (0,1)$, and $f_0 : X \to K$, where K segment [-1,+1] on the y-axis. The function defined by the formula $f_0(x) = \sin\frac{\pi}{2x}$, then it is absolutely clear that for any choice of the value $f_0(0)$ the continuation of this function f_0 on the closure $\overline{X} = H$ will not be continuous [1].

In such cases, there is the following general construction, which makes it possible to solve all such problems.

Let's do the following first.

Note. Let (Y, f) be some extension of the space X and let $\widetilde{X} = X \bigcup (Y \setminus f(X))$ be the disjoint union of the set textitX with lots of this extension's overgrowth. Consider the mapping $\varphi : \widetilde{X} \to Y$, which on X coincides with f, and on the subset $Y \setminus f(X)$ is the identity mapping, then it is clear that φ is a -bijection. Let $\widetilde{\tau}$ be the preimage of the topology of the space Y under the mapping φ then φ is a homeomorphism of the space $(\widetilde{X}, \widetilde{\tau})$ onto Y. It is easy to see that $\widetilde{\tau}$ induces on X the original topology and, in addition, the pair $(\widetilde{X}, i)$, where $i : X \subset \widetilde{X}$ is a X extension equivalent to the original one. Thus, it is always possible to replace a given extension (Y, f) of the space X by such an equivalent extension $(\widetilde{X}, i)$, in which X itself is a subspace.

We now describe this construction.

Now, using the above remark, we will apply the described construction to construct a continuous continuation of the function $f_0(x) = \sin\frac{\pi}{2x}$. It is clear that in this case it is best to take the compact set $H = [0,1]$ as $b(X)$, then by Z we denote the square $Z = [0,1] \times [-1,+1]$, $b_0(X)$ -combining the graph f_0 with the segment [-1,+1] of the y-axis, and $\bar{f}_0 : b_0(X) \to [-1,1]-$orthogonal projection onto the y-axis. Spaces $\hat{X}$ will be a *T*-shaped figure, which is a theoretical - multiple union of the segment [0,1] of the abscissa axis and the segment [0,1] of the ordinate axis, however, the topology of the space textit $\widetilde{X}$, of course, differs significantly from the topology induced from R^1. If $\varphi : \widetilde{X};\ b_0(X)$ is a homeomorphism, then the mapping $(\widetilde{f}_0 \circ \varphi) : \widetilde{X} \to [-1,1]$ is the required continuous extension of the function $f_0 : X \to [-1,1]$.

The given general construction of extension of mappings is based on the following:

Proposition 4.35. [1]. Let X be an arbitrary completely regular space of infinite weight, and $f_0 : X \to K$ be some continuous mapping into a compact set K. Then there exists a compact extension $b(X)$ whose weight does not exceed the maximum weights of X and K, to which f_0 can be continued continuously.

A mapping between *ANR* spaces is called a fine homotopy equivalence if there is an f -proper mapping "to" and for every open cover U of Y there is a proper mapping $g: Y \to X$, that $gf : X \to X$ is properly homotopic to the identity map by means of a homotopy bounded by an open cover

$$f^{-1}U = \{ f^{-1}(u) : u \in U \}$$

Let $F : A \times I \to X$ be some homotopy and U be an open cover of X. A homotopy is said to be bounded by the cover U if for any point $a \in A$ there exists an element U of the cover U containing the sets $F(\{a\} \times I)$.

In [4], there is the following

Proposition 10 [4]. Let $Y \subset X$ be an ANR space. In this case, the following are equivalent:

1) Y is homotopically dense in X;

2) the embedding $Y \subset X$ is a fine homotopy equivalence;

3) The set $C(Q,Y)$ is everywhere dense in $C(Q,X)$.

This Proposition 10 [4] and Theorem 4 [16] imply

Theorem 1. For any locally compact ANR space X its compactification αX holds:

a) X is homotopy dense in αX;

b) the embedding of X in is a fine homotopy equivalence;

c) the set $C(Q,X)$ is everywhere dense in $C(X,\alpha X)$;

Corollary 1. For any open subset $U \subset R$ of the straight line R, the compactification of αU holds U as homotopy dense. Hence the line R is homotopically dense in αR.

In [4], there is the following

Proposition 16 [4]. Let A be a homotopically dense subspace of X. A set A is ANR if and only if X is ANR.

From this fact, Theorem 1, and Corollary 1, we obtain

Theorems 2. For any locally compact X space, X has ANR if and only if the compactification of αX is an ANR space.

It follows from the definition of homotopy dense subsets of X that its complement is homotopically negligible (or negligible) in X.

In [4], there is the following

Theorem 1.4.4 [4]. Let X be ANR. For a closed subset $A \subset X$ the following conditions are equivalent:

a) A is a $Z-$set in X;

b) A is homotopically negligible in X;

From this fact, the definition of homotopically negligible subsets and Theorem 2 follow.

Corollary 2. For any local compact ANR space X, its outgrowth $EX = \alpha X \setminus X$ of the compactification αX is a $Z-$ set in αX.

Let M be a non-compact $Q-$manifold and αM its compactification. By Corollary 2, its growth FM is a $Z-$set in αM. In this case, by a result of [6], the compact set αM is a $Q-$manifold. So there is a place

Theorem 3. The compactification αM of any $Q-$manifold M is a Hilbert cube Q i.e. $\alpha M;\ Q$.

It is known that the product of a Hilbert cube Q and $[0,1)$ is a Q manifold.

Therefore, there is:

Corollary 3. The compactification of $\alpha(Q\times[0,1))$ is homeomorphic to Q. those. $\alpha(Q\times[0,1));\ Q$.

If $X-ANR$ is a space, then $X \times Q$ - is a Q manifold [6], therefore.

Corollary 4. For any non-compact ANR space X, the space $\alpha(X \times Q)$ is homeomorphic to the Hilbert cube Q. those. $\alpha(X \times Q);\ Q$.

We noted earlier that for any infinite compact set X the space $P(X)$ is homeomorphic to the Hilbert cube Q and for any non-empty closed subset $A\subset X, A\neq\varnothing$ other than $X, A\neq X$ subspace of $P(X)\setminus P(A)$ is a Q manifold. Further, we noted that for any $n\in N$ the spaces $P(X)\setminus P_n(X)$ and $P(X)\setminus P_{f,n}(X)$ also have Q- manifolds. Hence the space $P(X)\setminus P_f(X)$ is also a Q manifold. Since these subsets $P_n(X), P(A), P_{f,n}(X)$ are closed in $P(X)$. Also, if $A\subset X, A\neq\varnothing, A\neq X$ and $A-$ is open in X, then $S_P(A)$ is also $Q-$manifolds. In this case, it follows from Theorem 3 and Corollaries 3–4.

Theorem 4. For any infinite compact set X;

a) the compactification $\alpha(P(X)\setminus P(A))$ of the space $P(X)\setminus P(A)$ is homeomorphic to Q, where $A\subset X, A\neq X, \overline{A}=A$; those. $\alpha(P(X)\setminus P(A))$; Q;

b) $\alpha(S_P(A))$ is homeomorphic to Q, where $A\subset X, A\neq X$ A is open in X; i.e. $\alpha(S_P(A))$; Q;

c) $\alpha(P_{\nabla n}(X))$; Q, where $n\in N$, $P_{\nabla n}(X)=P(X)\setminus P_n(X)$;

d) $\alpha(P_{f,n}(X))$; Q, where $n\in N$, $P_{\nabla f,n}(X)=P(X)\setminus P_{f,n}(X)$;

e) $\alpha(P(X)\setminus P_f(X))$; Q.

It is known that any χ_0 –degrees of a non-one-point openly generated K space of all probability measures $P(K^{\chi_0})$ is homeomorphic to the Hilbert $Q=I^{\chi_0}$.

Therefore, the conclusion of Theorem 4 remains true, for any χ_0 degree of non-one-point openly – generated compact sets K.

It is known that if X is an infinite compact space, then the space $P_\omega(X)$ is $\sigma-$compact and the compact set $\alpha(P_\omega(X))$ is infinite-dimensional.

By Cayley's theorem, a convex compact set $\alpha(P_\omega(X))$ is homeomorphic to Q. It means there is a place.

Theorem 5. For any infinite compact set X, the compactification $\alpha(P_\omega(X))$ is homeomorphic to the Hilbert cube Q. those. $\alpha(P_\omega(X))$; Q.

If X is locally compact connected, then X is homotopically dense in εX [11].

A non-compact Q - manifold M admits compactification if there exists a compact Q - manifold $N\supset M$ such that $N\setminus M$ -is a Z-set in N [6].

If X is a ANR -compact space, $A\subset X$ is a Z-set such that $X\setminus A$ is a Q- manifolds, then X is also a Q–manifold [6].

If Z-sets are strictly negligible in X and $X\setminus Y$ is a countably sum of Z-sets, then X is homeomorphic to Y, where X - topologically complete separable metric space [6].

Lemma 1. Let A be a proper subset of an infinite compact X. Then for any $n\in N$ the subspace $P_n(A)$ is a $Z-$set in $P(X)$.

Proof. It is known that $\overline{P(A)}$; $P(X)$**,** if A is dense in X $P(A)\in AR$ since $P(X)$ is convex and locally convex. Let us show that $P_{\nabla n}(X)$ is homotopically dense in $P(A)$. Take the measure $\mu_0\in P(X)\setminus P_n(A)$ i.e. $\mu_0=m_1^0\delta_{x_1}+m_2^0\delta_{x_2}+...+m_{n+2}^0\delta_{x_{n+2}}$ where $\sum_{i=1}^{n+2}m_i^0=1$, $m_i^0>0, m_i\neq 0, x_i\in A$.

Let us construct a homotopy $h(\mu,t): P(X)\times[0,1]\to P(X)$ sloping $h(\mu,t)=t\mu_0+(1-t)\mu$.

The homotopy $h(\mu,t)$ satisfies the following conditions:

1. If $t=0$ then $h(\mu,0)=\mu$. those. $h(\mu,0)=id_{P(X)}$;

2. If $t=1$ then $h(\mu,1)=\mu_0$;

3. If $t>0$, then $h(\mu,t)\overline{\in} P_n(A)$, since $|\operatorname{supp} h(\mu,t)|\geq n+2$.

Hence the set $P_{\nabla n}(A)$ is homotopically dense in $P(X)$. In this case, the fact of its complement [4] implies that $P_n(A)$ is a $Z-$set in $P(X)$. Ω

Lemma 2. Let X be an infinite compact Y,X, $Y\neq X$ $\overline{Y}=X$ then $P(Y)$ is homotopy dense in $P(X)$.

Proof. It is known that the subspace $P(Y)$ is everywhere dense in $P(X)$ and $P(Y)$ is an AR space, since $P(Y)$ convex and locally convex.

Let $x_0\in X\setminus Y$. In this case the Dirac measure δ_{x_0} at this point belongs to $P(X)\setminus P(Y)$. those. $\delta_{x_0}\in P(X)\setminus P(Y)$. We construct a homotopy $h(\mu,t):P(X)\times[0,1]\to P(X)$ by setting

$$h(\mu,t)=t\delta_{x_0}+(1-t)\mu.$$

This homotopy $h(\mu,t)$ satisfies the following properties:

1. If $t=0$, then $h(\mu,0)=\mu$. those. $h(\mu,0)=id_{P(X)}$;

2. If $t=1$, then $\delta_{x_0}=h(\mu,1)\in P(Y)$;

3. If $t>0$, then $h(\mu,t)=t\delta_{x_0}+(1-t)\mu\overline{\in} P(X\setminus Y)$ since $|\operatorname{supp} h(\mu,t)|\bigcap X\setminus Y=\varnothing$ .i.e. for any $t\in(0,1]$ measure $h(\mu,(0,1])\in P(Y)$. This means that $P(Y)$ is homotopy dense in $P(X)$.Ω

Lemma 3. Let X be an infinite compact set and Y its own closed subset of X, then $P(Y)$ is a $Z-$set in $S_P(Y)$.

Proof. By convexity and local convexity, the set $S_P(Y)$ is an AR space and $S_P(Y)$ is everywhere dense in $P(X)$. Naturally, $P(Y)\subset S_P(Y)$, since the supports of the measure $\mu\in P(Y)$ lie entirely in Y.

Let us now show that $P(Y)$ is a $Z-$set in $S_P(Y)$.

We construct a homotopy $h(\mu,t):S_P(Y)\times[0,1]\to S_P(Y)$ by setting

$$h(\mu,t)=\mu_0 t+(1-t)\mu$$

where $\mu_0\in P(X)\setminus P(Y)$ i.e. $\operatorname{supp}\mu_0\subset X\setminus Y$ or $\operatorname{supp}\mu_0\bigcap Y=\varnothing$.

Now check that this homotopy satisfies the following:

1. If $t=0$, then $h(\mu,0)=\mu$. those. $h(\mu,0)=id_{S_P(Y)}$;

2. If $t=1$, then $h(\mu,1)=\mu_0\overline{\in} P(Y)$;

3. If $t>0$, then $h(\mu,t)\in S_P(Y)$, since the support of the measure $h(\mu,t)$ contains points in the set Y . those. $\operatorname{supp} h(\mu,t)\bigcap Y\neq\varnothing$.

In this case, by virtue of the result of Theorem 1.4.4 [4], the set $P(Y)$ is a $Z-$set in $S_P(Y)$. Note that in this case $S_P(Y)$ is homeomorphic to the Hilbert space ℓ_2 [4]. Ω

Theorem 6. Let X be an infinite compact $A_1,A_2,...,A_n,...$ closed subsets of X such that $\bigcup_{i=1}^{\infty}A_i=A$ is everywhere dense in X and A is not open in X. Then the $\bigcup_{i=1}^{\infty}S_P(A_i)$ of the manifold contains the homeomorphic manifold $\sigma \times S$.

Proof. In this case, from infinite and compacta we have $P(X);\ Q$ and we can assume that $A_i \subset A_{i+1}$ for each i. The everywhere density of A in X implies that $P(A)$ is everywhere dense in $P(X)$. Obviously, the subspace subspace $S_P(A_i)$ is also everywhere dense in $P(X)$ and $S_P(A_i)$ is homeomorphic to ℓ_2 for each i [11]. From $A_i \subset A_{i+1}$ it follows that $S_P(A_i) \subset S_P(A_{i+1})$. Obviously, the union $\bigcup_{i=1}^{\infty} S_P(A_i)$ is everywhere dense in $P(X)$. It follows from the results of [4] that $\bigcup_{i=1}^{\infty} S_P(A_i) \in AR$. Note that $P(A)$ is $\bigcup_{i=1}^{\infty} S_P(A_i) \subset S'_P(A)$. On the other hand, $P(A) - \sigma$ -compact by Lemma 3 is a $\sigma - Z -$set in $S_P(A)$.

In this case, the space $\bigcup_{i=1}^{\infty} S_P(A_i)$ satisfies the conditions of Theorem 5.3.9 [4]. those. $\bigcup_{n=1}^{\infty} S_P(A_i) \ni \sigma \times \ell_2;\ \sigma \times S$. Ω

Using the definitions, properties of strongly universal spaces, applying Theorems 5.3.10 [4] we have

Corollary 1. Under the conditions of Theorem 1, the space $\bigcup_{i=1}^{\infty} S_P(A_i)$ is strongly $M_0(\omega)$ universal.

We note that the pseudointerior S of the Hilbert cube Q is homeomorphic to ℓ_2. It was shown in [12] that the functor F taking Hilberts Q to Hilberts Q, with finite supports, takes the pseudointerior of S to the pseudointerior of S. those. if F is a functor such that $F(Q) \cong Q$ and F is a finitely supported functor then $(F(Q),F(S));\ (Q,S)$.

This implies that the functor $P_n(S);\ S;\ \ell_2$. In this case $P_\omega(S)$ is $\bigcup_{n=1}^{\infty} P_n(S)$ and $P_\omega(S) \in AR$. How Theorem 1 is proved is the following.

Corollary 2. For the pseudointerior S of the cube Q, the following holds: the manifold $P_\omega(S)$ contains a subset homeomorphic to the manifold $\sigma \times S$.

Corollary 3. Let X be an infinite compact set and $A_i \subset A_{i+1}$ be a sequence of closed subsets of the compact set X such that $\bigcup_{i=1}^{\infty} A_i$ is everywhere dense in X and $\bigcup_{i=1}^{\infty} A_i \neq X$. Then $P(\bigcup_{i=1}^{\infty} A_i) \setminus P_\omega(\bigcup A_i) \cong \sigma \times S$ if $P(\bigcup_{i=1}^{\infty} A_i)$ contains the Hilbert cube Q and $P_n(\bigcup_{i=1}^{\infty} A_i);\ \sigma$.

If X is an ANR space, then $X \times [0,1)$ is locally compact, then $\alpha(X \times [0,1)) \in AR$ [2].

Therefore, for the line R the space $\alpha(R \times [0,1))$ is AR compact.

If X is a metric compact set, then $K(X) = \alpha(X \times [0,1))$ is called a cone for X, where $X \times \{0\} \subset X \times [0,1)$.

Note that if X is a local compact metric space, then αX is a metric compact space.

For each $\sigma -$compact space X and functor P the space $P(X)$ is also $\sigma -$compact.

The main task (problem) is:

For any local compact metric space X and a covariant functor $F: Comp \to Comp$ that preserves locally compact spaces, is $F(\alpha X) \cong \alpha F(X)$?

Problem 15. § 12 in [4] is as follows.

Let K be a locally compact AR space and let $K_1 \subset K_2 \subset ... \subset K_n \subset ...$ be a growing sequence of compact subsets of K such that $K=\bigcup_{n=1}^{\infty}K_n$, $K_n \subset intK_{n+1}$, $n\in N$ and for each $n\in N$ subspace $K\setminus K_n$ is contractible. Then

a) K is homotopy dense in αK;
b) K is AR compact;
c) $\alpha K;\ Q$ if K is a $Q-$manifold.

From this fact and the locally compact $[0,1)\in AR$ it follows that $\alpha[0,1)$ is an AR compact i.e. $\alpha[0,1)\in AR$. On the other hand, the space $R\times[0,1)$ is locally compact, then $\alpha(R\times[0,1))\in AR$.

Hence, the one-point extension $\alpha(0,1)$ of the interval $(0,1)$ of the line is ANR compact, which is homeomorphic to the circle.

Therefore, the one-point extension of the line R in the plane is the projective line P_1 and is ANR compact (that is, the line P_1 with a point at infinity). If we take the interval $(0,1)$ its one-point Alexandrov extension, as if the points {0} and {1} are glued together at one point. those. points {0} and {1} as one point $\{0;1\}$ is added to the set $(0,1)$. In the case when X is a line $(-\infty,+\infty)$ the set of real numbers, two infinities $-\infty$ and $+\infty$ are glued together as a single point $\omega-$ at infinity i.e. $\alpha R=R\bigcup\{\omega\}$. Note that $\alpha R\in ANR$.

Theorem 7. Let X be a connected locally compact locally connected space, then X is homotopy dense in αX and the identity embedding $id_X : X\to\alpha X$ is a homotopy equivalence.

Theorem 8. For any infinite compact set X, the one-point extension $\alpha P_\omega(X)$ of the space $P_\omega(X)$ is homeomorphic to the Hilbert cube Q.

Proof. It is known that for an infinite contact X the space $P_\omega(X)$ is a subspace of $P(X)$ and $P_\omega(X)$ in compact AR is everywhere dense in $P(X)$. Obviously, $\alpha P_\omega(X)$ is an AR space (compact set), i.e. the compact set $\alpha P_\omega(X)$ is AR and from the theorem of Keller and Kelly the compact set $\alpha P_\omega(X)$ is homeomorphic to the Hilbert cube Q. We can assume that $\alpha P_\omega(X);\ P(X)$. Note that the subspace $P_\omega(X)$ is homotopy dense in $\alpha P_\omega(X)$. Ω

Corollary 4. For any non-compact $Q-$ manifold X, its one-point Alexandrov compactification αX is homeomorphic to the Hilbert cube Q. those. $\alpha X;\ Q$.

Corollary 5. For any infinite locally compact space X, $P(\alpha X)$ is homeomorphic to $P(X)$, i.e., $P(\alpha X)$ is homeomorphic to the Hilbert cube Q.

Let X be an infinite compact set and $A_1\subset A_2\subset ...\subset A_n\subset ...$ a sequence of closed sets of X such that $\bigcup_{i=1}^{\infty}A_i$ is everywhere dense in X and $\bigcup_{i=1}^{\infty}A_i\neq X$. Then $P(\bigcup_{i=1}^{\infty}A_i)=\bigcup_{i=1}^{\infty}P(A_i)$ and $P_\omega(\bigcup_{i=1}^{\infty}A_i)=\bigcup_{i=1}^{\infty}P_\omega(A_i)$. In this case, the results of [11] imply that the subspaces $P(\bigcup_{i=1}^{\infty}A_i)$ and $P_\omega(\bigcup_{i=1}^{\infty}A_i)$ are boundary sets of $P(X)$. If $P(\bigcup_{i=1}^{\infty}A_i)$ contains a Hilbert cube Q, then $P(\bigcup_{i=1}^{\infty}A_i);\ \sum$ [11]. On the other hand, if $\bigcup_{i=1}^{\infty}A_i$ is a countable set, then $P_\omega(A_i);\ \sigma$ [eleven]. those. the space $\bigcup_{i=1}^{\infty}P_\omega(A_i)$ is homeomorphic to σ. In this case, from Theorem 3.1 [13] we get that $P(\bigcup_{i=1}^{\infty}A_i)\setminus P_\omega(\bigcup_{i=1}^{\infty}A_i)$ is homeomorphic to $\sum\times S;\ \sigma\times S$.

So there is

Theorem 9. For any locally compact infinite space X, the space $\alpha P_{\omega}(\alpha X)$ is homeomorphic to the Hilbert cube Q.

For X a locally compact space X the following holds:

1) X is an ANR-compact, $A \subset X - Z -$set such that $X \setminus A - Q -$manifolds, then X is also a $Q -$manifold [6];

2) If $X \setminus X_0$ is vaguely in X and X_0 is $ANR \Rightarrow X$ is ANR [6];

3) If $M \times R^{\infty}$ is an R^{∞} manifold, if M is an open set in R^{∞} [2];

4) For Q^{∞} the manifold $M \times Q^{\infty}$; M [2];

5) If X is an $ANR-$space, then $\alpha(X \times [0,1)) \in AR$ i.e. $\alpha(X \times [0,1))$ is $ANR-$compact [6];

6) The Hilbert brick is an open (even monotone) image of a one-dimensional compactum. those. $f: X \to Q$, $\dim X = 1$, X compact set f is open (monotone) [2];

7) An arbitrary finite-dimensional ANR compact space is metrizable [2,8];

8) If X is an ANR compact and αX; $Q \Rightarrow X$ is a Q manifold [6];

9) For a locally compact connected and locally connected X αX; $Q \Leftrightarrow X - Q -$manifold [6];

10) If X is an infinite compact set and $Y \subset X$ is a discrete dense subset of $(P(X), P(Y))$; $(Q, \sum)$ [11];

11) The space X satisfies the $LCAP$-property (locally compact approximation property) [4] for each covering $U \in cov X$ there exists a mapping $f: X \to X$ such that $(f, id_X) < U$ and $C\ell_X f(X)$ locale is compact.

a) Any locally compact space $X \in LCAP$.

b) Any compact set K, $X \times K \in LCAP$ if $X \in LCAP$.

12) If X is ANR and $LCAP$, then any $Z-$set is a strongly $Z-$set [4];

13) If Z_{σ} is a subspace of $A \subset Q$, and $\sum \subset A$, then $(Q, \sum)$ [13];

14) If $A \subset S$ then (S, A); $(S, \sum)$ [13];

15) A space X is said to be contractible in ∞ denoted by $X \in C(E)$ if every $A \subset X$ $\exists$ is a compact $B \subset X$ such that every component of $X \setminus B$ contractible to $X \setminus A$ [14].

Theorem [14]. If $X \in ANR$ and $X \in \sum$ if X has a finite number of connected components. Then $X \in C(E) \Leftrightarrow F(X) \in ANR$ and EX is vaguely in $F(X)$.

Definition [15]. If X is a space and $Z' \subset Z$, then Z' is said to be vague in Z if the homotopy $H: Z \times I \to Z$ $T-$that $h(z,0) = z$ $H(z,t) \in Z'$, $\forall z \in Z, 0 < t \leq 1$.

Lemma 4.3 [16]. If Z' is vague in Z, then the embedding of $Z \setminus Z'$ in Z is a homotopy equivalence.

Theorem 4.4 [16]. $X \in ANR$, $X \in \sum$ and $X \in C(X)$ then $X \subset F(X)$ is a homotopy equivalence.

Theorem 1 [16]. A point a of a locally compact metric space M vaguely $\Leftrightarrow$ $\exists$ the homotopy $H \times M[0,1] \to M$ satisfies the conditions

a) $H(x,0) = x, \forall x \in M$.

b) $H(x,t) \neq a, \forall x \in M, t > 0$.

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