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PRODUCTS OF WREATHS AND CASCADES IN FINITE STATE MACHINES WITH LATTICE-VALUED BIPOLAR FUZZY COVERINGS

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Abstract

In this paper, we discuss lattice valued bipolar fuzzy finite state machines, homomorphisms, and weak coverings. In addition, two lattice-valued bipolar fuzzy finite state machines are examined in terms of their covering relations. It is discussed how direct products, cascade products, and wreath products are covered. Product machines exhibit a few transitive properties of covering relations. As a result, studying lattice-valued bipolar fuzzy finite state machines is an important step.

Keywords: Wreath product , Homomorphism , Cascade product ,Covering.

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1.Introduction

In [1], Zadeh proposed a fuzzy set theory. Fuzzy automatons were first conceptualized mathematically by Wee in [2]. Afterward, Atanassov introduced intuitionistic fuzzy sets [3, 4] that are highly useful for dealing with vagueness in a variety of higher order fuzzy sets. As a generalization of fuzzy finite state machines, Jun introduced intuitionistic fuzzy finite state machines [5–7] based on intuitionistic fuzzy sets. They also introduced intuitionistic successors, intuitionistic subsystems, intuitionistic submachines, and intuitionistic q-twins. According to Zhang and Li [8], intuitive fuzzy recognizers are based on intuition. Among the most important contributions of Atanassov to fuzzy sets was the lattice-valued intuitionistic fuzzy set theory. The theory of bipolar fuzzy finite State Machines was presented by [9] Young Bae Jun and Jacob Kavikumar. Followed by Bae Bipolar-valued Fuzzy Finite Switchboard State Machines was introduced by J. Kavikumar [10] .A lattice-valued bipolar fuzzy finite state machine has been shown to contain cascade products, homomorphisms, wreath products and weak coverings.

2 Preliminaries

Definition 2.1.A bipolar-valued fuzzy set φ in X is an object having the form $\varphi = \{(x, \varphi^N, \varphi^P) | \, x\epsilon X\}$ where $\varphi^N: X \to [-1, 0]$ and $\varphi^P: X \to [0, 1]$ are mappings, where $\varphi^P(x)$ denotes the positive membership degree and $\varphi^N(x)$ denotes the Negative membership degree. We shall use the notation $\langle \varphi^N, \varphi^P \rangle$ instead of $\varphi = \{(x, \varphi^N, \varphi^P) | \, x\epsilon X\}$

Definition 2.2. Let $W = (R, Y, B)$ is a lattice-valued bipolar fuzzy finite state machine(LB_{FSM}), where R and Y are finite nonempty sets, called the set of states and the set of input symbols, respectively, and φ = $\langle \varphi^N, \varphi^P \rangle$ is a bipolar fuzzy set in R × Y × R.

Let Y^* denote the set of all words of elements of Y of finite length. Let λ denote the empty word in Y^* and |y| denote the length of y for every y ∈ Y.

Definition 2.3. Suppose $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2, \ldots$ The complete direct product of W_1 and W_2 is denoted by $(R_1 \times R_2, Y_1 \times Y_2, B_1 \times B_2)$ where the Cartesian product of their states, input symbols are taken

$$\varphi^P_{B_1 \times B_2}((r_1, r_2), (y_1, y_2), (s_1, s_2)) = \varphi^P_{B_1}(r_1, y_1, s_1) \wedge \varphi^P_{B_2}(r_2, y_2, s_2),$$
$$\varphi^N_{B_1 \times B_2}\big((r_1, r_2), (y_1, y_2), (s_1, s_2)\big) = \varphi^N_{B_1}(r_1, y_1, s_1) \vee \varphi^N_{B_2}(r_2, y_2, s_2),$$

where $\varphi^P_{B_1 \times B_2}: (R_1 \times R_2) \times (Y_1 \times Y_2) \times (B_1 \times B_2) \to (0, 1]$, $\varphi^N_{B_1 \times B_2}: (R_1 \times R_2) \times (Y_1 \times Y_2) \times (B_1 \times B_2) \to [-1, 0), \forall \, (r_1, r_2), (s_1, s_2) \in R_1 \times R_2, (y_1, y_2) \in Y_1 \times Y_2$.

Definition 2.4. Suppose $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2.$ The restricted direct product of W_1 and W_2 is denoted by $W_1 \wedge W_2 = (R_1 \times R_2, Y, B_1 \times B_2)$, and

$$\varphi^P_{B_1 \wedge B_2}((r_1, r_2), b, (s_1, s_2)) = \varphi^P_{B_1}(r_1, b, s_1) \wedge \varphi^P_{B_2}(r_2, b, s_2),$$
$$\varphi^N_{B_1 \wedge B_2}\big((r_1, r_2), b, (s_1, s_2)\big) = \varphi^N_{B_1}(r_1, b, s_1) \vee \varphi^N_{B_2}(r_2, b, s_2),$$

where $\varphi^P_{B_1 \wedge B_2}: (R_1 \times R_2) \times Y \times (R_1 \times R_2) \to (0, 1]$, $\varphi^N_{B_1 \wedge B_2}: (R_1 \times R_2) \times Y \times (R_1 \times R_2) \to [-1, 0), \forall \, (r_1, r_2), (s_1, s_2) \in R_1 \times R_2, \forall \, b \in Y$.

Theorem 2.1. Suppose $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM} , where $j = 1, 2.$ The subsequent statements are true.

i) $W_1 \times W_2$ be a LB_{FSM}.

ii) $W_1 \wedge W_2$ be a LB_{FSM}, where $Y_1 = Y_2 = Y$.

Definition 2.5. If $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2$. A surjective partial function $\gamma: R_1 \to R_2$ and a function $\beta: Y_1 \to Y_2$ together form an ordered pair (γ, β) which is referred to as a covering of W_1 by W_2 denoted as $W_1 \leq W_2$ if
$\varphi^P_{B_1}\big(\gamma(s), y, \gamma(r)\big) \leq \varphi^P_{B_2}(s, \beta(y_1), r)$,$\varphi^N_{B_1}\big(\gamma(s), y, \gamma(r)\big) \leq \varphi^N_{B_2}(s, \beta(y_1), r)$,
$\forall\, y_1 \in Y$ and $r, s \in \gamma$.

Theorem 2.2. If $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2{,}3$. If $W_1 \leq W_2$ and $W_2 \leq W_3$ then $W_1 \leq W_3$.

Theorem 2.3.If $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2{,}3$. Then $W_1 \wedge W_2 \leq W_1 \times W_2$.

Theorem 2.4.If $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2{,}3$. If $W_1 \leq W_2$, then
1) $W_1 \times W_3 \leq W_2 \times W_3$ and $W_3 \times W_1 \leq W_3 \times W_2$
2) $W_1 \wedge W_3 \leq W_2 \wedge W_3$ and $W_3 \wedge W_1 \leq W_3 \wedge W_2$

Where $Y_1 = Y_2 = Y_3 = Y$.
Corollary 2.1.If $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2{,}3$. If $W_1 \leq W_2$, then
1) $W_1 \wedge W_3 \leq W_2 \times W_3$ where $Y_1 = Y_3 = Y$
2) $W_3 \wedge W_1 \leq W_3 \times W_2$ where $Y_1 = Y_3 = Y$

Corollary 2.2.If $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2{,}3{,}4$. If $W_1 \leq W_2$ and $W_3 \leq W_4$ then The subsequent statements are true.
1) $W_1 \times W_2 \leq W_2 \times W_4$,
2) $W_1 \wedge W_3 \leq W_2 \wedge W_4$ where $Y_1 = Y_2 = Y_3 = Y_4 = Y$
3) $W_1 \wedge W_3 \leq W_2 \times W_4$ where $Y_1 = Y_3 = Y$.

3. LB_{FSM} for wreath products and cascades

Definition 3.1.Let $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2$. The cascade product of $W_1$and W_2 is denoted by $W_1 \omega W_2 = (R_1 \times R_2, Y_2, B_1 \omega B_2)$ and

$$\varphi^P_{B_1 \omega B_2}((r_1, r_2), a, (s_1, s_2)) = \varphi^P_{B_1}(r_1, \omega(r_2, a), s_1) \wedge \varphi^P_{B_2}(r_2, a, s_2),$$
$$\varphi^N_{B_1 \omega B_2}\big((r_1, r_2), a, (s_1, s_2)\big) = \varphi^N_{B_1}(r_1, \omega(r_2, a), s_1) \vee \varphi^N_{B_2}(r_2, a, s_2),$$

Where $\varphi^P_{B_1 \omega B_2}: (R_1 \times R_2) \times Y_2 \times (R_1 \times R_2) \to (0, 1]$, $\varphi^N_{B_1 \omega B_2}: (R_1 \times R_2) \times Y_2 \times (R_1 \times R_2) \to [-1, 0)$, $\omega: R_2 \times Y_2 \to Y_1$ be a function, $\forall\, (r_1, r_2), (s_1, s_2) \in R_1 \times R_2, \forall\, a \in Y_2$.

Definition 3.2. Let $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2$. The wreath product of $W_1$and W_2 is denoted by
$W_1 \circ W_2 = (R_1 \times R_2, {Y_1}^{R_2} \times Y_2, B_1 \circ B_2)$ and

$$\varphi^P_{B_1 \circ B_2}((r_1, r_2), (d, a), (s_1, s_2)) = \varphi^P_{B_1}(r_1, d(r_2), s_1) \wedge \varphi^P_{B_2}(r_2, a, s_2),$$
$$\varphi^N_{B_1 \circ B_2}\big((r_1, r_2), (d, a), (s_1, s_2)\big) = \varphi^N_{B_1}(r_1, d(r_2), s_1) \vee \varphi^N_{B_2}(r_2, a, s_2),$$

where
$$\varphi^P_{B_1 \circ B_2}: \big(R_1 \times R_2\big) \times ({Y_1}^{R_2} \times Y_2) \times \big(R_1 \times R_2\big) \to (0, 1],$$
$$\varphi^N_{B_1 \circ B_2}: \big(R_1 \times R_2\big) \times ({Y_1}^{R_2} \times Y_2) \times \big(R_1 \times R_2\big) \to [-1, 0),$$
$${Y_1}^{R_2} = \{d \mid d: R_2 \to Y_1\},$$
$$\forall\, ((r_1, r_2), (d, a), (s_1, s_2) \in \big(R_1 \times R_2\big) \times ({Y_1}^{R_2} \times Y_2) \times \big(R_1 \times R_2\big).$$

Theorem 3.1. Let $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2$. Then

1) $W_1 \omega W_2$ be a LB_{FSM},
2) $W_1 \circ W_2$ be a LB_{FSM}.

Proof: Using the identical approach as Theorem 3.1. in the reference [10], it is simple to demonstrate the validity of the result.

4. Property of coverings

Definition 4.1.If $W_1 = (R_1, Y_1, B_1)$ and $W_2 = (R_2, Y_2, B_2)$ are two LB_{FSM}. An ordered pair (γ, β) is said to be a weak covering of W_1 by W_2 denoted by $W_1 \leq_\omega W_2$, if γ is a surjective partial function from R_2 to R_1 and β is a partial function from $Y_1$to Y_2.

$$\varphi^P_{B_1}\big(\gamma(s), y, \beta(r)\big) \leq \varphi^P_{B_2}(s, \beta(y_1), r),$$
$$\varphi^N_{B_1}\big(\gamma(s), y, \beta(r)\big) \geq \varphi^N_{B_2}(s, \beta(y_1), r),$$

$\forall\, y_1 \in Y$ and $r, s \in \gamma$.

Only the fact that in Definition 4.1. is a partial function and in a Definition 2.5. is a function that separates a weak covering from a covering. As a result, every covering is inadequate.

Definition 4.2. If $W_1 = (R_1, Y_1, B_1)$ and $W_2 = (R_2, Y_2, B_2)$ are two LB_{FSM}. An ordered pair (δ, σ) is said to be a homomorphism where $\delta: R_1 \to R_2$ and $Y_1 \to Y_2$

$$\varphi^P_{B_1}(r, b, s) \leq \varphi^P_{B_2}\big(\delta(r), \sigma(b), \delta(s)\big),$$

$$\varphi^N_{B_2}(r, b, s) \leq \varphi^N_{B_2}\big(\delta(r), \sigma(b), \delta(s)\big),$$

$\forall\, r, s \in R_1$, and $\forall\, b \in Y_1$.

The ordered pair (δ, σ) is said to be strong homomorphism, if

$$\varphi^P_{B_2}\big(\delta(r), \sigma(b), \delta(s)\big) = \vee \left\{\varphi^P_{B_1}(r, b, u) \middle| u \in R_1, \delta(u) = \delta(s)\right\}$$

$$\varphi^N_{B_2}\big(\delta(r), \sigma(b), \delta(s)\big) = \wedge \left\{\varphi^N_{B_1}(r, b, u) \middle| u \in R_1, \delta(u) = \delta(s)\right\}$$

$\forall\, r, s \in R_1$, and $\forall\, b \in Y_1$.

An isomorphism (also known as strong isomorphism) (δ, σ) between W_1 and W_2 is referred to as a homomorphism(also known as strong homomorphism) if both δ and σ are bijective.

Theorem 4.1. Suppose $W_1 = (R_1, Y_1, B_1)$ and $W_2 = (R_2, Y_2, B_2)$ are LB_{FSM}. Consider a homomorphism $(\delta, \sigma): W_1 \to W_2$. If (δ, σ) is a one-to-one strong homomorphism, then

$$\varphi^P_{B_2}\big(\delta(r), \sigma(\, y_1), \delta(s)\big) = \varphi^P_{B_1}(r,\, y_1, s),$$

$$\varphi^N_{B_2}\big(\delta(r), \sigma(\, y_1), \delta(s)\big) = \varphi^N_{B_1}(r,\, y_1, s)\,,$$

$\forall\, r, s \in R_1$, and $\forall\, b \in Y_1$.

Proof:

As (δ, σ) are strong homomorphism, it follows that

$$\varphi^P_{B_2}\big(\delta(r), \sigma(y_1), \delta(s)\big) = \vee \left\{\varphi^P_{B_1}(r, y_1, u) \middle| u \in R_1, \delta(u) = \delta(s)\right\}$$

$$\varphi^N_{B_2}\big(\delta(r), \sigma(y_1), \delta(s)\big) = \wedge \left\{\varphi^N_{B_1}(r, y_1, u) \middle| u \in R_1, \delta(u) = \delta(s)\right\}$$

As δ is an injective function and $\delta(u) = \delta(s)$, it follows that $u = s$. Therefore,

$$\varphi^P_{B_2}\big(\delta(r), \sigma(y_1), \delta(s)\big) = \varphi^P_{B_1}(r, y, u)$$

$$\varphi^N_{B_2}\big(\delta(r), \sigma(y_1), \delta(s)\big) = \varphi^N_{B_1}(r, y, u).$$

Theorem 4.2. Suppose $W_1 = (R_1, Y_1, B_1)$ and $W_2 = (R_2, Y_2, B_2)$ are LB_{FSM}. Consider a homomorphism $(\delta, \sigma): W_1 \to W_2$.

1) If (δ, σ) is a surjective strong homomorphism and δ is a injective, then $W_2 \leq W_1$,
2) If δ is injective, then $W_1 \leq W_2$.

Proof:

1) As (δ, σ) are surjective strong homomorphism we can conclude that there are surjective functions $\delta: R_1 \to R_2$ and $\sigma: Y_1 \to Y_2$. We can define $\gamma: R_1 \to R_2$ and $\beta: Y_1 \to Y_2$. Since σ is a surjective function, there must be at least one original image a in R_1 such that $\sigma(b) = b'$ for some b' in R_2. We can then define $\beta(b') = b$. If (δ, σ) is a strong homomorphism with δ being one to one, then
$$\varphi^P_{B_2}\big(\delta(r), \sigma(b), \delta(s)\big) = \varphi^P_{B_1}(r, b, s)$$
$$\varphi^N_{B_2}\big(\delta(r), \sigma(b), \delta(s)\big) = \varphi^N_{B_1}(r, b, s)$$
$$\forall\, r, s \in R_1, \text{ and } \forall\, b' \in Y_2.$$
If $\beta(b') = b$, then
$$\varphi^P_{B_2}\big(\gamma(r), b', \gamma(s)\big) = \varphi^P_{B_2}(\delta(r), \sigma(b), \delta(s))$$
$$\varphi^P_{B_1}(r, b, s) = \varphi^P_{B_1}(r, \beta(b'), s)$$
$$\varphi^N_{B_2}\big(\gamma(r), b', \gamma(s)\big) = \varphi^N_{B_2}(\delta(r), \sigma(b), \delta(s))$$
$$\varphi^N_{B_1}(r, b, s) = \varphi^N_{B_1}(r, \beta(b'), s).$$
Therefore (γ, β) is a covering of W_2 by W_1, $W_2 \leq W_1$.
2) Since $(\delta, \sigma): W_1 \to W_2$ be a homomorphism, there exists a mapping $\delta: R_1 \to R_2$ and $\sigma: Y_1 \to Y_2$, such that
$$\varphi^P_{B_1}(r_1, b_1, s_1) \leq \varphi^P_{B_2}(\delta(r_1), \sigma(b_1), \delta(s_1))$$
$$\varphi^N_{B_1}(r_1, b_1, s_1) \leq \varphi^N_{B_2}(\delta(r_1), \sigma(b_1), \delta(s_1))$$
$$\forall\, r_1, s_1 \in R_1, \text{ and } \forall\, b_1 \in Y_1.$$
Suppose $\gamma: R_2 \to R_1$. If $\delta(r_1) = r_2$ then $\gamma(r_2) = r_1$. As δ is one-to-one function, we can infer the r_1 is uniquely determined. Therefore, γ is surjective partial function. Let $\beta: Y_1 \to Y_2$, $\beta = \sigma$, then
$$\varphi^P_{B_1}(\gamma(r_2), b_1, \gamma(s_2)) \leq \varphi^P_{B_2}(r_2, \beta(b_1), s_2)$$
$$\varphi^N_{B_1}(\gamma(r_2), b_1, \gamma(s_2)) \geq \varphi^N_{B_2}(r_2, \beta(b_1), s_2)$$
Therefore (γ, β) is a covering of W_1 by W_2, $W_1 \leq W_2$.

Corollary 4.1. Suppose $W_1 = (R_1, Y_1, B_1)$ and $W_2 = (R_2, Y_2, B_2)$ are LB_{FSM}. Consider a homomorphism $(\delta, \sigma): W_1 \to W_2$. Then

1) If (δ, σ) is a strong homomorphism and δ is a bijective, then $W_2 \leq_\omega W_1$,
2) If δ is injective, then $W_1 \leq_\omega W_2$.

Proof:

1) The evidence corresponds to that of theorem 4.2(1).
2) By utilizing theorem 4.2(2), we can determine $W_1 \leq W_2$. As all covering are weak coverings, it follows that $W_1 \leq_\omega W_2$.

Theorem 4.3. Suppose $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2, 3$. If $W_1 \leq_\omega W_2$ and $W_2 \leq_\omega W_3$ then $W_1 \leq_\omega W_3$.
Proof:
Given $W_1 \leq_\omega W_2$, we can conclude that there is a partial surjective function $\gamma: R_2 \to R_1$ and partial function $\beta: Y_1 \to Y_2$ which satisfy the following condition.

$$\varphi^P_{B_1}(\gamma(s_1), y_1, \gamma_1(r_1)) \leq \varphi^P_{B_2}(s_1, \beta_1(y_1), r_1),$$
$$\varphi^N_{B_1}(\gamma(s_1), y_1, \gamma_1(r_1)) \geq \varphi^N_{B_2}(s_1, \beta_1(y_1), r_1),$$

for every y_1 there is a member of the domain β_1, and every s_1 and r_1 that are members of the domain of γ_1.

If $W_2 \leq_\omega W_3$, we can conclude that there is a partial surjective function $\gamma_2: R_3 \to R_2$ and a partial function $\beta_2: Y_2 \to Y_3$, which satisfies the following condition

$$\varphi^P_{B_2}(\gamma_2(s_2), y_2, \gamma_2(r_2)) \leq \varphi^P_{B_3}(s_2, \beta_2(y_2), r_2),$$
$$\varphi^N_{B_2}(\gamma_2(s_2), y_2, \gamma_2(r_2)) \geq \varphi^N_{B_3}(s_2, \beta_2(y_2), r_2),$$

for every y_2 there is a member of the domain β_2, and every s_2 and r_2 that are members of the domain of γ_2.

Let $\gamma = \gamma_1 \circ \gamma_2: R_3 \to R_1$, $\beta = \beta_2 \circ \beta_1: Y_1 \to Y_3$. It is evident that γ is a partial function with surjective properties and β is also partial function. If there exists Y_1such that it belongs to the domain of β and β, and there exist s and r which belong to the domain of γ and γ_2 respectively, then

$$\begin{aligned}
\varphi^P_{B_1}\big(\gamma(s), y_1, \gamma(r)\big) &= \varphi^P_{B_1}(\gamma_1 \circ \gamma_2(s), y_1, \gamma_1 \circ \gamma_2(s)) \\
&= \varphi^P_{B_1}(\gamma_1(\gamma_2(s), y_1, \gamma_1(\gamma_2(r)) \\
&\leq \varphi^P_{B_2}(\gamma_2(s), \beta_1(y_1), \gamma_2(r)) \\
&\leq \varphi^P_{B_3}(s, \beta_2(\beta_1(y_1)), r) \\
&= \varphi^P_{B_3}(s, \beta_2 \circ \beta_1(y_1), r\ ) \\
&= \varphi^P_{B_3}(s, \beta(y_1), r).
\end{aligned}$$

Alike, we can demonstrate that $\varphi^N_{B_1}\big(\gamma(s), y_1, \gamma(r)\big)$ is not less than $\varphi^N_{B_3}(s, \beta(y_1), r)$. It is evident that (γ, β) confirms to the necessary conditions for a weak covering of W_1 by W_3.

Theorem 4.4. Let $W_j = (R_j, Y_j, B_j)$ is a LB_{FSM}, where $j = 1, 2$. Then

1) $W_1 \omega W_2 \leq W_1 \circ W_2$
2) $W_1 \circ W_2 \leq W_1 \times W_2$
3) $W_1 \omega W_2 \leq W_1 \times W_2$

Proof:
Define the function $\omega_c: R_2 \to Y_1$ as follows $\omega_c(s_2) = \omega(s_2, c), \forall\, s_2 \in R_2$ and $c \in Y_2$.

1) Let $\beta: Y_2 \to Y_1^{R_2} \times Y_2$ be defined as $\beta(c) = (\omega_c, c)$ and let γ be the identity map on $R_1 \times R_2$.
2) Let $\beta: Y_1^{R_2} \times Y_2 \to Y_1 \times Y_2$, by $\beta(d, c) = (d(s_2), c)$, while γ denotes the identity map on $R_1 \times R_2$
3) Given that $W_1 \omega W_2 \leq W_1 \circ W_2$ and $W_1 \circ W_2 \leq W_1 \times W_2$, it follows from Theorem 2.2. that $W_1 \omega W_2 \leq W_1 \times W_2$.

Theorem 4.5. If $W_i = (R_i, Y_i, B_i)$ is a LB_{FSM}, where $i = 1, 2,3$. If $W_1 * W_2$, then

1) If $\omega_1: R_3 \times Y_3 \to Y_1$, is provided, there is a $\omega_2: R_3 \times Y_3 \to Y_2$ that satisfies $W_1 \omega W_3 \leq W_2 \omega_2 W_3$. If (γ, β) is a cover of $W_1$by W_2 and β is onto, then for every $\omega_1: R_1 \times Y_1 \to Y_3$, there exist $\omega_2: R_2 \times Y_2 \to Y_3$ such that $W_3 \omega_1 W_1 \leq W_3 \omega_2 W_2$.
2) $W_1 \circ W_3 \leq W_2 \circ W_3$ and $W_3 \circ W_1 \leq W_3 \circ W_2$.

Proof:
Given $W_1 * W_2$ we can conclude the existence of a partial function $\gamma: R_2 \to R_1$ which is surjective, and a function $\beta: Y_1 \to Y_2$,

$$\varphi_{B_1}^{P}(\gamma_1(s_2), y_1, \gamma_1(r_2)) \leq \varphi_{B_2}^{P}(s_2, \beta(y_1), r_2),$$
$$\varphi_{B_1}^{N}(\gamma_1(s_2), y_1, \gamma_1(r_2)) \geq \varphi_{B_1}^{N}(s_2, \beta(y_1), r_2),$$

for every y_1 there is a member of the domain y_1, and every s_2 and r_2 that are members of the domain of γ_1

1) Let $\omega_1: R_3 \times Y_3 \to Y_1$, set $\omega_2 = \beta_1 \circ \omega_1$and β_2 as an identity mapping on Y_3. Define $\gamma_2: R_2 \times R_3 \to R_1 \times R_3$ by $\gamma_2((r_2, r_3)) = (\gamma_2(r_2), r_3)$. It is evident that (γ_2, β_2) satisfies the condition for covering, $W_1 \omega_1 W_2 \leq W_2 \omega_2 W_3$. Now, let $\omega_1: R_1 \times Y_1 \to Y_3$, set $\omega_2: R_2 \times Y_2 \to Y_3$ such that $\omega_2\big(r_2, \beta_1(y_1)\big) = \omega_1(\gamma_1(r_2), y_1)$. Since β_1 is onto and $y_1$is finite, such a ω_2 exists. However it is not unique. Define $\gamma: R_3 \times R_2 \to R_3 \times R_1$ by $\gamma_2((r_3, r_2)) = (\ r_3, \gamma_1(r_2)\ )$ and set $\beta_2 = \beta_1$. It is obvious that (γ_2, β_2) satisfies the conditional for a covering, $W_3 \omega_1 W_1 \leq W_3 \omega_2 W_2$.
2) The function $\gamma_2: R_2 \times R_3 \to R_1 \times R_3$ can be defined as $\gamma_2((r_2, r_3)) = (\gamma_1(r_2), r_3)$ and the function $\beta_2: Y_1^{R_3} \times Y_3 \to Y_2^{R_3} \times Y_3$ can be defined as $\beta_2(d, y_3) = (\beta_1 \circ d, y_3)$. It is evident that γ_2 is a partial function that covers all values and β_2 is a complete function. Another function $\gamma_2: R_3 \times R_2 \to R_3 \times R_1$ can be defined as $\gamma_2((r_3, r_2)) = (\ r_3, \gamma_1(r_2)\ )$ and the function $\beta_2: Y_3^{R_1} \times Y_1 \to Y_3^{R_2} \times Y_2$ by

$\beta_2(d, y_1) = (d \circ \gamma_1, \beta_1(y_1))$. It is apparent that γ_2 is a partial function that covers all the values and β_2 is also a partial function.

Corollary 4.2. If $W_i = (R_i, Y_i, B_i)$ is a LB_{FSM}, where $i = 1, 2,3,4$. If $W_1 \leq W_2$ and $W_3 \leq W_4$, then

1) $W_1 \circ W_2 \leq_\omega W_2 \circ W_4$
2) $W_1 \omega W_3 \leq_\omega W_2 \circ W_4$
3) $W_1 \circ W_2 \leq W_2 \times W_4$
4) $W_1 \omega W_3 \leq W_2 \times W_4$.

Proof:
Using Theorem 4.5. and 4.3., we are able to demonstrate that

$W_1 \circ W_3 \leq_\omega W_2 \circ W_4$. Equally, we can establish the validity of (2), (3) and (4).

5.Conclusion

In automata theory, product is one of the most fundamental operations. The present study delves into the multiplication of finite state machines and coverings that are equipped with lattice-valued bipolar fuzzy attributes.

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