Idempotent graphs corresponding to *R*-strongly unit regular Monoids

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Abstract

If S is a monoid with G as group of units then we can define the relation R_G on S as follows; x $R_G y$ if $x = yu$ for some $u \in G$.A unit regular monoid S is said to be R-strongly unit regular if $R = R_G$ where R is Green's equivalence on S. In this paper we generate a weighted directed graph namely W corresponding to the set of idempotents of S and we have studied about the properties of weights associated with the edges of W .

Keywords: Equivalences, Unit regular monoids, idempotents .

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Preliminaries

Throughout this paper let S be a finite R-strongly unit regular semigroup and let $E(S)$ denote the set of idempotents of S. The vertices of the graph W are idempotents of S .For $e, f \in E(S)$,e is adjacent to f if there is an element u belonging to the group of units G of S such that $e = fu$. Let $G(e,f) = \{u \in G \,; e = fu\}$. Let $W(e,f)$ denote the cardinality of the set $G(e,f)$ and we define $W(e,f)$ as the weight of the edge connecting e and f. We define W as the idempotent graph (weighted symmetric directed graph with loops)corresponding to the R-strongly unit regular monoid. Let $V(W)$ denote the vertex set of the graph W and let $E(W)$denote the edge set of the graph.

A monoid S is said to be unit regular if for each element $s \in S$ there exists an element u in the group of units G of S such that $s = sus$. If S is a monoid with G as group of units then we can define the relations R_G and L_G *on* S as follows.

$$xR_G y \leftrightarrow x = yu \text{ for some } u \in G.$$

$$xL_G y \leftrightarrow x = uy \text{ for some } u \in G.$$

Then R_G and L_G are equivalences on S. Let $E = E(S)$ denote the set of idempotent of the regular monoid S and $R = \{(x,y) \in S \times S: xS = yS\}, L = \{(x,y) \in S \times S: Sx = Sy\}$, be the Green's equivalences on S. A unit regular monoid is said to be R-strongly unit regular if $R = R_G$on S. [5]

The study of the full transformation semigroup consisting of all self maps of a non empty finite set X, has given rise to interesting results. If X denote the finite set $\{1,2,\dots,n\}$, then $T(X)$ denote the semigroup (under the composition of mappings , where for $\alpha, \beta \in T(X), \alpha\beta$ means α acts first) of all mappings from X to X. The symmetric group $G(X)$ consisting of all permutations of X is a subgroup of $T(X)$. The study of $T(X)$ has in particular led to many results of combinatorial nature. $T(X)$ has the property that for $\alpha, \beta \in T(X), \alpha R\beta$ implies $\alpha = \beta\gamma$, where $\gamma \in G(X)$. Thus $T(X)$ is R-strongly unit regular.

Let $G(e,e)$ be denoted as $G(e)$.Then we have the following propositions.

Proposition 1.1. [5]Let S be a monoid. For $e, f \in E(S), G(e) = G(f)$, if eLf .

Proposition 1.2.[5] Suppose S is a monoid. Then for $e, f. g \in E(S), G(e,f)G(f,g) = G(e,g)$.

Proposition 1.3.[5]Let G denote the group of units of the monoid S. Then for every $v \in G, vG(e,f)v^{-1} = G(vev^{-1}, vfv^{-1})$.

Proposition 1.4.[5] Let $\alpha R\beta$ in $T(X)$. Then $|G(\alpha,\beta)| = (n-m)!$ where $|X\alpha| = |X\beta| = m$ and $|X| = n$.

Proposition 1.5.[5]Let X be a finite set of cardinality equal to n. Then each R-class of $T(X)$ corresponding to a partition $n = n_1 + n_2 + \cdots + n_m$ of n contains $n_1 n_{2\dots} n_m$ idempotents.

Proposition 1.6.[5]Let S be a monoid with group of units G. Then for every $x \in S, G(x) = \{u \in G: xu = x\}$ is a subgroup of G. Further if $x, y \in S$ and $xw = y$ for some $w \in G$, then $G(x,y) = \{u \in G: xu = y\}$ is a right coset of $G(x)$ and a left coset of $G(y)$.

Properties of weight associated with the edges of *W*

Proposition 2.1. let S be a finite R-strongly unit regular semigroup and let W be the weighted directed graph corresponding to the idempotents of S. For $e, f \in V(W), W(e,f) = W(f,e)$.

Proof: $e = fu$ if and only if $eu^{-1} = f$. So $u \in G(e,f)$ if and only if $u^{-1} \in G(f,e)$. Hence $|G(e,f)| = |G(f,e)|$. That is $W(e,f) = W(f,e)$.

Proposition 2.2. Let W be the idempotent graph of the R-strongly unit regular monoid. Then for $e, f \in V(W), W(e,e) = W(f,f)$ if eLf.

Proof: Since $W(e,e), W(f,f)$ are defined as the cardinality of the sets $G(e), G(f)$ the result follows from the Proposition 1.1.

An edge connecting the vertices e and f in the graph W is denoted as *ef*

Proposition 2.3. For $e, f. g \in V(W)$, if ef and $fg \in E(W)$, then $eg \in E(W)$. Also in this case we have $W(e,f) = W(f,g) = W(e,g)$

Proof : By Proposition 1.2 , if eR_Gf and fR_Gg then eR_Gg Therefore if ef and $fg \in E(W)$, then $eg \in E(W)$. By Proposition 1.6 we have $|G(e)| = |G(f)| = |G(g)| = |G(e,f)| = |G(f,g)| = |G(e,g)|$. Hence $W(e,f) = W(f,g) = W(e,g)$.

Proposition 2.4. Let *W* be the idempotent graph of the *R*-strongly unit regular monoid and let *G* denote the group of units of the monoid *S*. Then for every $v \in G$ and $ef \in E(W)$, $W(e,f) = W(vev^{-1}, vfv^{-1})$.

Proof: By Proposition 1.3, for $ef \in E(W)$, $|vG(e,f)v^{-1}| = |G(vev^{-1}, vfv^{-1})|$. Hence $W(e,f) = W(vev^{-1}, vfv^{-1})$, since the number of elements in $vG(e,f)v^{-1}$ is same as $W(e,f)$.

Proposition 2.5. Let *W* be the idempotent graph of the *R*-strongly unit regular monoid .Then each component of *W* is a complete graph with weight of each edge being equal in a component and the number of components of *W* is same as the number of *R*-classes of S.

Proof: Since *S* is a *R*-strongly unit regular monoid , for any two idempotents *e*, *f* in the same $R-$ class,$G(e.f)$ is non empty. Hence *e* is adjacent to *f*. Since $E(S) = V(W)$ and $|G(e,f)| = W(e,f) = |G(e)|$ the result follows.

Proposition 2.6. If *S* is an inverse unit regular monoid then the idempotent graph corresponding to *S* is a null graph with only loops.

Proof : Each *R*-class of an inverse unit regular monoid contains only one idempotent**,** the result follows. Also $G(e)$ is non empty since the identity element belongs to $G(e)$.

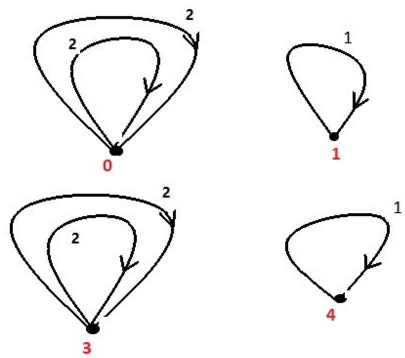
Now we will study about idempotent graphs corresponding to the full transformation semigroup $T(X)$. The next proposition gives the weight of each edge in each component of the idempotent graph corresponding to the full transformation semigroup.

Proposition 2.7. The number of vertices in each component of the idempotent graph corresponding to the full transformation semigroup $T(X)$ is $n_1 n_{2\ldots} n_m$ where $n = n_1 + n_2 + \cdots + n_m$ is a partition of $n = |X|$. Also for $\epsilon_1, \epsilon_2 \in E\big(T(X)\big)$,where $E\big(T(X)\big)$ is the set of idempotents in $T(X)$,$W(\epsilon_1, \epsilon_2) = (n-m)!$, where $|X\epsilon_1| = m$ and $|X| = n$.

Proof : The result follows from proposition 1.4 and 1.5.

Example 2.8. As an illustration of Proposition 2.3, 2.5 we have the following example. Let $S = \{0,1,2,3,4,5\}$ .Then *S* is a *R*-strongly unit regular monoid with respect to multiplication modulo 6, since every element of *S* can be written as $x = eu$, where $e \in E$ and $u \in G$. For example $2 = 4 \times_6 5$ .Here the set of group of units of *S* is $G = \{1,5\}$ and set *E* of idempotents of S is $\{0,1,3,4\}$.$G(0) = \{1,5\}, G(1) = \{1\}, G(3) = \{1,5\}, G(4) = \{1\}$

Fig 1



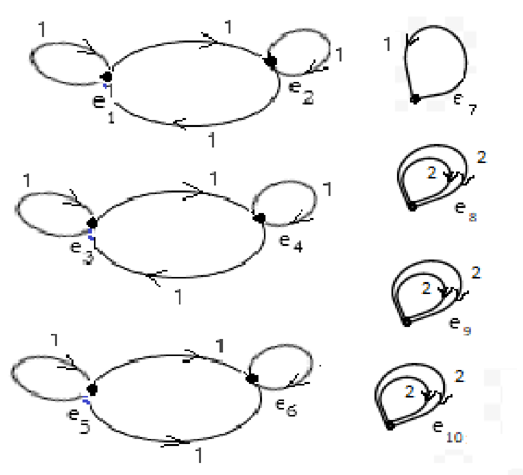
Example 2.9. Consider the full transformation semigroup $T(X)$ on the set $X = \{1,2,3\}$. Also let $G(X)$ denote the symmetric group on the set X. $G(X) =$

$$\left\{ f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

$$E(T(X)) = \left\{ e_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, e_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, e_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, e_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, e_7 = f_1, e_8 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, e_9 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, e_{10} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \right\}.$$

We have $e_1 f_2 = e_2$; $e_3 f_3 = e_4$; $e_5 f_6 = e_6$;$e_8 f_2 = e_8$; $e_9 f_6 = e_9$; $e_{10} f_3 = e_{10}$.

Fig 2



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