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Section H-Research paper

Fuzzy Laplace-Adomian Decomposition Method for Approximating Solutions of Time Fractional Klein-Gordan Equations in a Fuzzy Environment

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ABSTRACT

In this study, Authors proposed a new computational method for approximating the solution of linear and nonlinear time fractional Klein-Gordan equations in a fuzzy environment. Our approach, known as the Fuzzy Laplace-Adomian Decomposition method, involves using the Fuzzy Laplace Transform and an iterative process with Caputo fractional derivative to solve the equations with appropriate initial conditions. Our results demonstrate that this method effectively solves complex equations in a fuzzy environment and could help solve challenging problems in engineering and physics. Future research could focus on exploring the method's applicability to other types of equations in different fields.

Keywords: Fractional derivatives and integrals, Fuzzy number Fuzzy partial differential equations, Fuzzy Laplace Transform, Fuzzy Fractional Klein- Gordan equation.

1.0 Introduction

In recent years, the investigation of various mathematical equations involving fractional and fuzzy derivatives has gained significant attention.[1]–[3] These equations are encountered in diverse disciplines, including physics, engineering, biology, and finance.[4] Developing accurate and efficient numerical methods to solve such equations is crucial in comprehending complex phenomena and making reliable predictions. Mazandarani and Xiu (2021) conducted a comprehensive review of fuzzy differential equations, discussing various approaches and techniques for solving them and their applications in different fields.[1] Al-Sawalha, Amir, Shah, and Yar (2022) proposed a novel iterative transform method for analyzing fuzzy fractional Emden-Fowler equations.[5] Their method offered a new approach to efficiently solve these equations, contributing to the understanding of fuzzy systems. Arfan, Shah, Abdeljawad, and Hammouch (2021) presented an efficient tool for solving two-dimensional fuzzy fractional-ordered heat equations.[6] By utilizing numerical methods for partial differential equations, they provided a useful tool for analyzing and solving such equations. The paper uses the Caputo fractional derivative to present a new variational iteration method (VIM) for solving fractional convection-diffusion equations in large domains.[7]–[9] The method involves iteratively refining an initial approximation using the variational iteration technique. A study focuses on obtaining explicit solutions for the nonlinear Chen-Lee-Liu equation through an analytical approach.[10]

The study investigates the approximate boundary condition (ABC) method for fuzzy fractional differential equations using the Mittag-Leffler kernel differential operator.[11]–[13] The authors establish theorems regarding solution existence and uniqueness and provide illustrative examples to demonstrate the proposed approach's application. For the purpose of resolving a linked nonlinear system of Klein-Gordon equations, the researchers suggest the variational iteration technique (VIM).[14]–[16] The method involves constructing an initial approximation and iteratively refining it using the variational iteration technique, demonstrating effectiveness and accuracy through applied solutions. A residual series representation algorithm[17] is introduced for solving fuzzy Duffing oscillator equations. The

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method employs the residual power series[18] approach to approximate the solutions, showcasing its effectiveness and accuracy through numerical examples. In recent decades there has appeared a big number of nonlinear evolution equations.[7], [19]–[21] Shah et al. implemented the Adomian decomposition technique (ADM) for the fuzzy heat equation.[22] For fuzzy heat equations, a new approximation method has been introduced by Jameel et al. and Sitthiwirattham.[23], [24] To solve partial differential equations, Stepnicka and Valasek were the first to employ a fuzzy transform (PDEs).[25] Finally, Jameel studied the heat equation's semianalytical[26] solution in a fuzzy environment.

This work develops a method for approximating nonlinear PDE solutions.[27] Laplace Adomian Decomposition Method (LADM) combines ADM and Laplace transformations successfully employed this strategy.[28]–[30] Adomian's[31]–[35] ADM has been used in biology, physics, and chemical processes. The method provides a convergent series that may be computed quickly.[22] To solve the differential equation, applying the Laplace transformation and decomposing nonlinear terms into Adomian polynomials generates a recursive, iterative technique.

Al-Lehaibi and Baleanu et al. (2020) proposed modeling epidemic childhood diseases[36] using the Caputo-Fabrizio derivative and the Laplace Adomian decomposition method.[37] Similarly, Fatoorehchi and Alidadi (2017) utilized the extended Laplace transform method to analyze the Thomas-Fermi equation.[38] Kumar and Rani (2019) introduced the Laplace Adomian decomposition method for studying chemical ion transport in the soil.[39] Kumar and Umesh (2022) provided a comprehensive review of the recent developments in the Adomian decomposition method.[40] In Fuzzy Sets and Systems, Lupulescu and O'Regan (2021) explored calculus in quasilinear metric spaces and presented a novel derivative idea for set-valued and fuzzy-valued functions.[41], [42] In Computational and Applied Mathematics, Maitama and Zhao (2021) used the Homotopy analysis Shehu transform method to resolve fuzzy differential equations.[21], [43]

The Fuzzy Fractional Klein-Gordan equation[14], [44]

$$\frac{\partial^\alpha \tilde{\Xi}(\flat,\wp;\delta)}{\partial \wp^\alpha} \oplus \frac{\partial^2 \tilde{\Xi}(\flat,\wp;\delta)}{\partial \flat^2} \oplus a_1 \odot \tilde{\Xi}(\flat,\wp;\delta) \oplus b_1 \odot \tilde{\Xi}^2(\flat,\wp;\delta) \oplus c_1 \odot \tilde{\Xi}^3(\flat,\wp;\delta) = \tilde{k}_s(\delta)\Xi(\flat,\wp;\delta) \quad 1$$

subject to the fuzzy initial condition as

$$\tilde{\Xi}(\flat,0;\delta) = \tilde{\Xi}_0 \quad \text{and} \quad \frac{\partial \tilde{\Xi}(\flat,0;\delta)}{\partial \wp} = \tilde{\Xi}_1 \qquad 2$$

where $\Xi(\flat,\wp;\delta)$ be function of fuzzy number value and α is a parameter that describes the order of the fractional derivatives in Caputo sense. a_1, b_1, c_1 are constants, $\tilde{k}_s(\delta)$ is a fuzzy number.

2.0 Important terminology:

Definition 1: A fuzzy membership function of bounded support $\tilde{\Omega}_*: \mathbb{R} \to [0,1]$ is called a fuzzy number containing in $\mathbb{R}$ with a upper semi-continuous, normal, convex.[3]

Definition 2: Let $\tilde{\Omega}_* \in E^1$ and a fuzzy number $\tilde{\Omega}_*$ has the parametric form $\tilde{\Omega}_*(\delta) = (\underline{\Omega_*}(\delta), \overline{\Omega_*}(\delta))$ of functions $\underline{\Omega_*}(\delta), \overline{\Omega_*}(\delta)$, $0 \le r \le 1$ if and only if following conditions holds[45]:

1. $\underline{\Omega_*}(\delta), \overline{\Omega_*}(\delta)$ is left continuous function on $(0,1]$, bounded and right continuous at 0 with respect to δ.
2. $\underline{\Omega_*}(\delta), \overline{\Omega_*}(\delta)$ is a non-decreasing and non-increasing respectively.
3. for $0 \le \delta \le 1$, $\underline{\Omega_*}(\delta) \le \overline{\Omega_*}(\delta)$.

if $\delta = \underline{\Omega_*}(\delta) = \overline{\Omega_*}(\delta)$ then δ is called **crisp number.**

Definition 3: Let $\tilde{\Omega}_* \in E^1$ and for any $\delta \in [0,1]$. The following properties are true for all $\delta -$ level set of $\tilde{\Omega}_*(\delta) = \left[\underline{\Omega_*}(\delta), \overline{\Omega_*}(\delta)\right]$, $\tilde{\mho}_*(\delta) = \left[\underline{\mho_*}(\delta), \overline{\mho_*}(\delta)\right]$ is the crisp set, and scalar k, the interval based fuzzy arithematic is as,[3]

1. $\underline{\Omega_*}(\delta) = \underline{\mho_*}(\delta)$ and $\overline{\Omega_*}(\delta) = \overline{\mho_*}(\delta)$ is necessary for $\tilde{\Omega}_*(\delta) = \tilde{\mho}_*(\delta)$
2. $\tilde{\Omega}_* + \tilde{\mho}_* = [\underline{\Omega_*}(\delta) + \underline{\mho_*}(\delta), \overline{\Omega_*}(\delta) + \overline{\mho_*}(\delta)]$

3. $k \odot \widetilde{\Omega}_*(\delta) = \begin{cases} \left[k\underline{\Omega_*}(\delta), k\overline{\Omega_*}(\delta)\right] & k \geq 0, \\ \left[k\overline{\Omega_*}(\delta), k\underline{\Omega_*}(\delta)\right] & k < 0." \end{cases}$

Definition 4: The distance $D_T(\widetilde{\Omega}_*, \widetilde{\mho}_*)$ between two fuzzy numbers $\widetilde{\Omega}_*$ and $\widetilde{\mho}_*$ is defiend as follow, $D_T: E^1 \times E^1 \rightarrow R_+ \cup \{0\}$ by[3], [46]

$$D_T(\widetilde{\Omega}_*, \widetilde{\mho}_*) = \sup_{r\in[0,1]} d_H(\widetilde{\Omega}_*(\delta), \widetilde{\mho}_*(\delta)), \qquad 3$$

where,

$$d_H(\widetilde{\Omega}_*(\delta), \widetilde{\mho}_*(\delta)) = max\begin{Bmatrix} \left|\underline{\Omega_*}(\delta) - \underline{\mho_*}(\delta)\right|, \\ \left|\overline{\Omega_*}(\delta) - \overline{\mho_*}(\delta)\right| \end{Bmatrix} \qquad 4$$

is the Hausdroff distance between $\widetilde{\Omega}_*(r)$ and $\widetilde{\mho}_*(r)$. Thus, D_T is a metric space and has the following properties:

1. $D_T\left(\widetilde{\Omega}_*(\delta) \oplus \tilde{z}(\delta), \widetilde{\mho}_*(\delta) \oplus \tilde{z}(\delta)\right)$
   $= D_T(\widetilde{\Omega}_*, \widetilde{\mho}_*), \forall \widetilde{\Omega}_*, \widetilde{\mho}_*, \tilde{z} \in E^1,$
2. $D_T\left(k \odot \widetilde{\Omega}_*(\delta), k \odot \widetilde{\mho}_*(\delta)\right)$
   $= |k| D_T(\widetilde{\Omega}_*, \widetilde{\mho}_*), \forall k \in \mathbb{R}, \widetilde{\Omega}_*, \widetilde{\mho}_* \in E^1,$
3. $D_T(\widetilde{\Omega}_* \oplus \widetilde{\mho}_*, \tilde{z} \oplus \widetilde{w})$
   $\leq D_T(\widetilde{\Omega}_*, \tilde{z}) + D_T(\widetilde{\mho}_*, \widetilde{w}), \forall \widetilde{\Omega}_*, \widetilde{\mho}_*, \tilde{z}, \widetilde{w} \in E^1,$
4. (D_T, E^1) is a complete metric space.

Definition 5: Let $\widetilde{\Omega}_*, \widetilde{\mho}_* \in E^1$. If $\widetilde{\Omega}_* = \widetilde{\mho}_* + \tilde{z}$ such that there exists $\tilde{z} \in E^1$, $\tilde{z}$ is the Hukuhara difference of $\widetilde{\Omega}_*$ and $\widetilde{\mho}_*$, and is denoted by $\tilde{z} = \widetilde{\Omega}_* \ominus \widetilde{\mho}_*$. Note that $\widetilde{\Omega}_* + (-1)\widetilde{\mho}_* \neq \widetilde{\Omega}_* \ominus \widetilde{\mho}_*$.[3], [47]

Definition 6: A fuzzy valued function $\tilde{\Xi}: D_T \rightarrow E^1$ is called continuous at $(\Omega_0, \mho_0) \in D_T$ if $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $d(\tilde{\Xi}(\Omega, \mho), \tilde{\Xi}(\Omega_0, \mho_0)) < \epsilon$ whenever $|\Omega - \Omega_0| + |\mho - \mho_0| < \delta$. $\tilde{\Xi}$ is continuous on D_T, if $\tilde{\Xi}$ be continuous $\forall (\Omega, \mho) \in D_T$.[3], [47]

Definition 7: The order $\alpha > 0$ Riemann-Liouville integral operator $\mathcal{J}^\alpha$ is defined as[45], [48], [49]

$$\mathcal{J}^\alpha \Xi(\wp) = \frac{1}{\Gamma(\alpha)} \int_0^\wp (\wp - \mu)^{\alpha-1} \Xi(\mu) d\mu, \quad \wp > 0. \qquad 5$$

In the case of the fuzzy number-valued function $\tilde{\Xi}$, the R-L fractional integral of order α can be represented as follows:

$$[\mathcal{J}^\alpha \tilde{\Xi}(\wp; \delta)] = [\mathcal{J}^\alpha \underline{\Xi}(\wp; \delta), \mathcal{J}^\alpha \overline{\Xi}(\wp; \delta)], \quad \wp > 0, \qquad 6$$

Where,

$$\mathcal{J}^\alpha \underline{\Xi}(\wp) = \frac{1}{\Gamma(\alpha)} \int_0^\wp (\wp - \mu)^{\alpha-1} \underline{\Xi}(\mu) d\mu, \quad \wp > 0,$$

$$\mathcal{J}^\alpha \overline{\Xi}(\wp) = \frac{1}{\Gamma(\alpha)} \int_0^\wp (\wp - \mu)^{\alpha-1} \overline{\Xi}(\mu) d\mu, \quad \wp > 0.$$

Definition 8: The Caputo sense fractional derivative of $\Xi(\wp)$ is defined as follows[45], [48]–[50]:

$${}^c\mathcal{D}_\wp^\alpha g(\wp) = \mathcal{J}^{\zeta-\alpha} \mathcal{D}_\wp^\zeta g(\wp)$$

$$= \begin{cases} \frac{1}{\Gamma(\zeta-\alpha)} \int_0^\wp \frac{g^\zeta(\mu) d\mu}{(\wp-\mu)^{\alpha+1-\zeta}} & \zeta - 1 < \alpha < \zeta, \zeta \in \mathbb{N} \\ \frac{d^\zeta}{d\wp^\zeta} g(\wp) & \alpha = \zeta, \quad \zeta \in \mathbb{N} \end{cases} \qquad 7$$

Definition 9: Let $[\tilde{\Xi}(\wp; \delta)] = [\underline{\Xi}(\wp; \delta), \overline{\Xi}(\wp; \delta)]$ be a parametric representation of fuzzy valued function $\tilde{\Xi}(\wp; \delta)$, where $\delta \in [0,1], 0 < \alpha < 1$ and $\wp \in (a, b)$ as follow [45], [48]

1. If $\tilde{\Xi}(\wp; \delta)$ is a Caputo-type fuzzy fractional differential function in the first form can be expressed in the following way:

$$[{}^c\mathcal{D}_\wp^\alpha \tilde{\Xi}(\wp; \delta)] = [{}^c\mathcal{D}_\wp^\alpha \underline{\Xi}(\wp; \delta), {}^c\mathcal{D}_\wp^\alpha \overline{\Xi}(\wp; \delta)].$$

2. If $\tilde{\Xi}(\wp; \delta)$ is a Caputo-type fuzzy fractional differential function in the second form can be expressed in the following way:

$$[{}^c\mathcal{D}_\wp^\alpha \tilde{\Xi}(\wp; \delta)] = [{}^c\mathcal{D}_\wp^\alpha \overline{\Xi}(\wp; \delta), {}^c\mathcal{D}_\wp^\alpha \underline{\Xi}(\wp; \delta)].$$

where,

$${}^c\mathcal{D}_\wp^\alpha \underline{\Xi}(\wp; \delta) = \frac{1}{\Gamma(\zeta - \alpha)} \int_0^\wp \frac{\underline{\Xi}^\zeta(\mu) d\mu}{(\wp - \mu)^{\alpha+1-\zeta}},$$

$$\zeta - 1 < \alpha < \zeta, \quad \zeta \in \mathbb{N}$$

$$ {}^{c}\mathcal{D}_{\wp}^{\alpha}\tilde{\Xi}(\wp;\delta) = \frac{1}{\Gamma(\zeta-\alpha)}\int_0^{\wp} \frac{\tilde{\Xi}^{\zeta}(\mu)d\mu}{(\wp-\mu)^{\alpha+1-\zeta}}, $$

$$ \zeta-1<\alpha<\zeta, \quad \zeta\in\mathbb{N} $$

Definition 10: The Laplace transform of fuzzy function g is expressed as[45], [48], [51]

$$ \Xi(p) = \mathfrak{L}[\tilde{\Xi}(\wp;\delta)](p) \quad 8 $$

$$ = \int_0^{\infty} Exp(-p\wp)\odot\tilde{\Xi}(\wp)d\wp, \quad p>0 \text{ and integer.} $$

Definition 11: The transform Fuzzy Laplace for Caputo Type Fuzzy Fractional Derivative is[45]

$$ \mathfrak{L}[{}^{c}\mathcal{D}_{\wp}^{\alpha}\tilde{\Xi}(\wp;\delta)](p) = p^{\alpha}\Xi(p) - \sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\tilde{\Xi}^{q}(\flat,0;\delta), \quad \alpha\in(\zeta-1,\zeta] \quad 9 $$

Definition 12: The Mittag-Leffler operator $E_{\alpha}(\wp)$ is[34], [52]

$$ E_{\alpha}(\wp) = \sum_{n=0}^{\infty}\frac{\wp^{n}}{\Gamma(1+n\alpha)}, \quad 10 $$

where, $\alpha>0$.

3.0 Process of Fuzzy Laplace Adomian Decomposition Method (FLADM)

For a rough estimate, see the conceptual model in this part.

An Adomian decomposition[31], [32], [53] technique is used in conjunction with the fractional Caputo derivative

$$ \mathfrak{L}[{}^{c}\mathcal{D}_{\wp}^{\alpha}\tilde{\Xi}(\flat,\wp;\delta)] = \mathfrak{L}\,[\mathcal{D}_{\flat}^{2}\tilde{\Xi}(\flat,\wp;\delta) + a\tilde{\Xi}(\flat,\wp;\delta) + b\tilde{\Xi}^{2}(\flat,\wp;\delta) + c\tilde{\Xi}^{3}(\flat,\wp;\delta)] + \mathfrak{L}[\tilde{k}(\delta)\Xi(\flat,\wp;\delta)] \quad 11 $$

where $\alpha\in(\zeta-1,\zeta]$; therefore the Laplace Transform of Eq. 11 is

$$ \mathfrak{L}[\tilde{\Xi}(\flat,\wp;\delta)] - \sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\tilde{\Xi}^{q}(\flat,0;\delta) = $$

$$ \mathfrak{L}\,[\mathcal{D}_{\flat}^{2}\tilde{\Xi}(\flat,\wp;\delta) + a\tilde{\Xi}(\flat,\wp;\delta) + b\tilde{\Xi}^{2}(\flat,\wp;\delta) + c\tilde{\Xi}^{3}(\flat,\wp;\delta)] + \mathfrak{L}[\tilde{k}(\delta)\Xi(\flat,\wp;\delta)] \quad 12 $$

$$ \mathfrak{L}[\tilde{\Xi}(\flat,\wp;\delta)] = \left\{ \begin{array}{c} \frac{1}{p^{\alpha}}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\tilde{\Xi}^{q}(\flat,0;\delta) + \frac{1}{p^{\alpha}}\mathfrak{L}[\mathcal{D}_{\flat}^{2}\tilde{\Xi}(\flat,\wp;\delta) \\ +a\tilde{\Xi}(\flat,\wp;\delta) + b\tilde{\Xi}^{2}(\flat,\wp;\delta) + c\tilde{\Xi}^{3}(\flat,\wp;\delta)] \\ +\frac{1}{p^{\alpha}}\mathfrak{L}[\tilde{k}(\delta)\Xi(\flat,\wp;\delta)] \end{array} \right\} \quad 13 $$

(13)

Apply inverse Laplace Transform to Eq.13 then obtain,

$$ \tilde{\Xi}(\flat,\wp;\delta) = \mathfrak{L}^{-1}\left\{\frac{1}{p^{\alpha}}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\tilde{\Xi}^{q}(\flat,0;\delta) + \frac{1}{p^{\alpha}}\mathfrak{L}[\tilde{k}(\delta)\Xi(\flat,\wp;\delta)]\right\} + \mathfrak{L}^{-1}\left\{\frac{1}{p^{\alpha}}\mathfrak{L}[\mathcal{D}_{\flat}^{2}\tilde{\Xi}(\flat,\wp;\delta) + a\tilde{\Xi}(\flat,\wp;\delta) + b\tilde{\Xi}^{2}(\flat,\wp;\delta) + c\tilde{\Xi}^{3}(\flat,\wp;\delta)]\right\} \quad 14 $$

Suppose series solution of function $\tilde{\Xi}(\flat,\wp;\delta)$ as

$$ \tilde{\Xi}(\flat,\wp;\delta) = \sum_{n=0}^{\infty}\tilde{\Xi}_{n}(\flat,\wp;\delta) \quad 15 $$

Here, note that $\tilde{\Xi}(\flat,\wp;\delta)$ is a fuzzy number-valued function that can be represented in terms of δ -cut representations.

$$ \tilde{\Xi}(\flat,\wp;\delta) = [\underline{\Xi}(\flat,\wp;\delta), \overline{\Xi}(\flat,\wp;\delta)] \quad 16 $$

and the operator L is linear;

$$ \therefore \quad L\left(\sum_{n=0}^{\infty}\tilde{\Xi}_{n}(\flat,\wp;\delta)\right) = \sum_{n=0}^{\infty} L[\tilde{\Xi}_{n}(\flat,\wp;\delta)] \quad 17 $$

and the operator N is nonlinear; then Eq.14 becomes,

$$\sum_{n=0}^{\infty} \tilde{\Xi}_n(\flat, \wp; \delta) = \mathfrak{L}^{-1}\left\{\begin{array}{c}\frac{1}{p^\alpha}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\tilde{\Xi}^q(\flat, 0; \delta) \\ +\frac{1}{p^\alpha}\mathfrak{L}\left[\tilde{k}(\delta)\Xi(\flat, \wp; \delta)\right]\end{array}\right\}$$

$$+\mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2 \sum_{n=0}^{\infty}\tilde{\Xi}_n(\flat, \wp; \delta) + a\sum_{n=0}^{\infty}\tilde{\Xi}_n(\flat, \wp; \delta) + b\sum_{n=0}^{\infty}\tilde{\Xi}_n^2(\flat, \wp; \delta) + c\sum_{n=0}^{\infty}\tilde{\Xi}_n^3(\flat, \wp; \delta)]\right\}$$

18

and the fuzzy nonlinear term $N(\tilde{\Xi}) = a\tilde{\Xi}_n(\flat, \wp; \delta) + b\tilde{\Xi}^2(\flat, \wp; \delta) + c\tilde{\Xi}^3(\flat, \wp; \delta)$ is decomposed as follow:

$$\begin{aligned} N(\underline{\Xi}) &= \textstyle\sum_{n=0}^{\infty} \underline{A}_n \\ N(\overline{\Xi}) &= \textstyle\sum_{n=0}^{\infty} \overline{A}_n \end{aligned} \qquad 19$$

where $\tilde{A}_n = [\underline{A}_n, \overline{A}_n]$ is the set polynomials of Adomian. The $\tilde{A}_n$'s has the formula

$$\begin{aligned} \underline{A}_n &= \frac{1}{n!}\frac{\partial^n}{\partial\Xi^n}[N(\textstyle\sum_{i=0}^{n}\Xi^i\underline{\Xi}_i)]_{\Xi=0}, \quad n = 0,1,2,3,\ldots. \\ \overline{A}_n &= \frac{1}{n!}\frac{\partial^n}{\partial\Xi^n}[N(\textstyle\sum_{i=0}^{n}\Xi^i\overline{\Xi}_i)]_{\Xi=0}, \quad n = 0,1,2,3,\ldots. \end{aligned} \qquad 20$$

some $\tilde{A}_i, (i \to n)$ terms are as follow:

$$\begin{aligned} \underline{A}_0 &= N(\underline{\Xi}_0) \\ \underline{A}_1 &= \underline{\Xi}_1(N)(\underline{\Xi}_0) \\ \underline{A}_2 &= \frac{1}{2}(N)(\underline{\Xi}_0)\underline{\Xi}_1^2 + \underline{\Xi}_2(N)(\underline{\Xi}_0) \\ \underline{A}_3 &= \frac{1}{6}(N)(\underline{\Xi}_0)\underline{\Xi}_1^3 + \underline{\Xi}_2(N)(\underline{\Xi}_0)\underline{\Xi}_1 + \underline{\Xi}_3(N)(\underline{\Xi}_0) \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} \overline{A}_0 &= N(\overline{\Xi}_0) \\ \overline{A}_1 &= \overline{\Xi}_1(N)(\overline{\Xi}_0) \\ \overline{A}_2 &= \frac{1}{2}(N)(\overline{\Xi}_0)\overline{\Xi}_1^2 + \overline{\Xi}_2(N)(\overline{\Xi}_0) \\ \overline{A}_3 &= \frac{1}{6}(N)(\overline{\Xi}_0)\overline{\Xi}_1^3 + \overline{\Xi}_2(N)(\overline{\Xi}_0)\overline{\Xi}_1 + \overline{\Xi}_3(N)(\overline{\Xi}_0) \\ &\vdots \end{aligned}$$

substituting Eq.19 and Eq.20 into Eq.18 then obtaining parametric form of $\tilde{\Xi}(\flat, \wp; \delta)$ as follows:

$$\underline{\Xi}(\flat, \wp; \delta) = \sum_{n=0}^{\infty}\underline{\Xi}_n(\flat, \wp; \delta) = \begin{pmatrix} \mathfrak{L}^{-1}\{\frac{1}{p^\alpha}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\underline{\Xi}^q(\flat, 0; \delta) + \frac{1}{p^\alpha}\mathfrak{L}[\underline{k}(\delta)\Xi(\flat, \wp; \delta)]\} \\ +\mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\sum_{n=0}^{\infty}\underline{\Xi}_n(\flat, \wp; \delta) + \frac{1}{p^\alpha}\mathfrak{L}[\sum_{n=0}^{\infty}\underline{A}_n]\right\} \end{pmatrix}$$

$$\overline{\Xi}(\flat, \wp; \delta) = \sum_{n=0}^{\infty}\overline{\Xi}_n(\flat, \wp; \delta) = \begin{pmatrix} \mathfrak{L}^{-1}\{\frac{1}{p^\alpha}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\overline{\Xi}^q(\flat, 0; \delta) + \frac{1}{p^\alpha}\mathfrak{L}[\overline{k}(\delta)\Xi(\flat, \wp; \delta)]\} \\ +\mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\sum_{n=0}^{\infty}\overline{\Xi}_n(\flat, \wp; \delta) + \frac{1}{p^\alpha}\mathfrak{L}[\sum_{n=0}^{\infty}\overline{A}_n]\right\} \end{pmatrix}$$

21

apply the iterative technique to find the recursive equation in the following form,

$$\begin{aligned}
\underline{\Xi}_0(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\{\frac{1}{p^\alpha}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\underline{\Xi}^q(\flat,0;\delta) + \frac{1}{p^\alpha}\mathfrak{L}[\underline{k}(\delta)\Xi(\flat,\wp;\delta)]\}\\
\underline{\Xi}_1(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\,\underline{\Xi}_0(\flat,\wp;\delta)] + \frac{1}{p^\alpha}\mathfrak{L}[\underline{A}_0]\right\}\\
\underline{\Xi}_2(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\,\underline{\Xi}_1(\flat,\wp;\delta)] + \frac{1}{p^\alpha}\mathfrak{L}[\underline{A}_1]\right\}\\
&\vdots\\
\underline{\Xi}_{n+1}(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\,\underline{\Xi}_n(\flat,\wp;\delta)] + \frac{1}{p^\alpha}\mathfrak{L}[\underline{A}_n]\right\}
\end{aligned} \quad 22$$

$$\begin{aligned}
\overline{\Xi}_0(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\{\frac{1}{p^\alpha}\sum_{q=0}^{\zeta-1} p^{\alpha-q-1}\overline{\Xi}^q(\flat,0;\delta) + \frac{1}{p^\alpha}\mathfrak{L}[\overline{k}(\delta)\Xi(\flat,\wp;\delta)]\}\\
\overline{\Xi}_1(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\,\overline{\Xi}_0(\flat,\wp;\delta)] + \frac{1}{p^\alpha}\mathfrak{L}[\overline{A}_0]\right\}\\
\overline{\Xi}_2(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\,\overline{\Xi}_1(\flat,\wp;\delta)] + \frac{1}{p^\alpha}\mathfrak{L}[\overline{A}_1]\right\}\\
&\vdots\\
\overline{\Xi}_{n+1}(\flat,\wp;\delta) &= \mathfrak{L}^{-1}\left\{\frac{1}{p^\alpha}\mathfrak{L}[\mathcal{D}_\flat^2\,\overline{\Xi}_n(\flat,\wp;\delta)] + \frac{1}{p^\alpha}\mathfrak{L}[\overline{A}_n]\right\}
\end{aligned} \quad 23$$

Thus, the solution becomes

$$\underline{\Xi}(\flat,\wp;\delta) = \underline{\Xi}_0(\flat,\wp;\delta) + \underline{\Xi}_1(\flat,\wp;\delta) + \cdots \quad 24$$

$$\overline{\Xi}(\flat,\wp;\delta) = \overline{\Xi}_0(\flat,\wp;\delta) + \overline{\Xi}_1(\flat,\wp;\delta) + \cdots \quad 25$$

4.0 Convergence and Error Analysis:

Theorem 1: Let $\tilde{\Xi}_p(\flat,\wp;\delta)$ and $\tilde{\Xi}_n(\flat,\wp;\delta)$ be the members of Banach space H, and the exact solution of (1.1) be $\tilde{\Xi}(\flat,\wp;\delta)$.[54] The Series solution $\sum_{p=0}^{\infty}\tilde{\Xi}_p(\flat,\wp;\delta)$ converges to $\tilde{\Xi}(\flat,\wp;\delta)$, if $\tilde{\Xi}_p(\flat,\wp;\delta) \le \lambda\tilde{\Xi}_{p-1}(\flat,\wp;\delta)$ for $\lambda \in (0,1)$, that is for any $\Xi > 0, \exists E$ such that $||\tilde{\Xi}_{p+n}(\flat,\wp;\delta)|| \le \Xi, \forall p, n > E$.

Proof. Let

$$u_p(\flat,\wp;\delta) = \tilde{\Xi}_0(\flat,\wp;\delta) + \tilde{\Xi}_1(\flat,\wp;\delta) + \tilde{\Xi}_2(\flat,\wp;\delta) + \cdots + \tilde{\Xi}_p(\flat,\wp;\delta)$$

be the sequence of p^{th} partial sum of series $\sum_{p=0}^{\infty}\tilde{\Xi}_p(\flat,\wp;\delta)$. Now consider

$$\begin{aligned}
&\left|\left|u_{p+1}(\flat,\wp;\delta) - u_p(\flat,\wp;\delta)\right|\right|\\
&= ||\tilde{\Xi}_{p+1}(\flat,\wp;\delta)||\\
&\le \lambda||\tilde{\Xi}_p(\flat,\wp;\delta)||\\
&\le \lambda^2||\tilde{\Xi}_{p-1}(\flat,\wp;\delta)||\\
&\le \lambda^3||\tilde{\Xi}_{p-2}(\flat,\wp;\delta)||\\
&\vdots\\
&\le \lambda^{p+1}||\tilde{\Xi}_0(\flat,\wp;\delta)||.
\end{aligned} \quad 26$$

for $\forall n, p \in E$

Consider,

$$\begin{aligned}
&||u_p(\flat,\wp;\delta)-u_n(\flat,\wp;\delta)|| \\
&=||\tilde{\Xi}_{p+n}(\flat,\wp;\delta)|| \\
&=||(u_p(\flat,\wp;\delta)-u_{p-1}(\flat,\wp;\delta)) \\
&+(u_{p-1}(\flat,\wp;\delta)-u_{p-2}(\flat,\wp;\delta)) \\
&+(u_{p-2}(\flat,\wp;\delta)-u_{p-3}(\flat,\wp;\delta)) \\
&+\cdots+(u_{n+1}(\flat,\wp;\delta)-u_n(\flat,\wp;\delta))|| \\
&\le ||(u_p(\flat,\wp;\delta)-u_{p-1}(\flat,\wp;\delta))|| \\
&+||(u_{p-1}(\flat,\wp;\delta)-u_{p-2}(\flat,\wp;\delta))|| \\
&+||(u_{p-2}(\flat,\wp;\delta)-u_{p-3}(\flat,\wp;\delta))|| \\
&+\cdots+||(u_{n+1}(\flat,\wp;\delta)-u_n(\flat,\wp;\delta))|| \\
&\le \lambda^p||\tilde{\Xi}_0(\flat,\wp;\delta)|| \\
&+\lambda^{p-1}||\tilde{\Xi}_0(\flat,\wp;\delta)|| \\
&+\lambda^{p-2}||\tilde{\Xi}_0(\flat,\wp;\delta)|| \\
&+\cdots+\lambda^{p-1}||\tilde{\Xi}_0(\flat,\wp;\delta)|| \\
&=||\tilde{\Xi}_0(\flat,\wp;\delta)||(\lambda^p+\lambda^{p-1}+\cdots+\lambda^{p+1}) \\
&=\left|\left|\tilde{\Xi}_0(\flat,\wp;\delta)\right|\right|\left(\frac{1-\lambda^{p-n}}{1-\lambda}\right)\lambda^{n+1}.
\end{aligned} \quad 27$$

Since $0<\lambda<1$, and $\tilde{\Xi}_0(\flat,\wp;\delta)$ is bounded, so assume that,

$$\Xi=||\tilde{\Xi}_0(\flat,\wp;\delta)||(\frac{1-\lambda^{p-n}}{1-\lambda})\lambda^{n+1},$$

getting the desired result.

Also $\sum_{p=0}^{\infty}\tilde{\Xi}_p(\flat,\wp;\delta)$ is a cauchy sequence in H, which imples that there exists $\tilde{\Xi}_0(\flat,\wp;\delta)\in H$ such that $\lim_{p\to\infty}\tilde{\Xi}_p(\flat,\wp;\delta)=\tilde{\Xi}(\flat,\wp;\delta)$, Hence prove.

Theorem 2: Let $\sum_{p=0}^{q}\tilde{\Xi}_p(\flat,\wp;\delta)$ be the finite and approximate solution of $\tilde{\Xi}(\flat,\wp;\delta)$. If $||\tilde{\Xi}_{p+1}(\flat,\wp;\delta)||\le\lambda||\tilde{\Xi}_0(\flat,\wp;\delta)||$ for $\lambda\in(0,1)$, then the maximum absolute error is

$$||\tilde{\Xi}(\flat,\wp;\delta)-\sum_{p=0}^{q}\tilde{\Xi}_p(\flat,\wp;\delta)|| \le\frac{\lambda^{q+1}}{1-\lambda}||\tilde{\Xi}_0(\flat,\wp;\delta)||.$$

Proof.

$$\begin{aligned}
&||\tilde{\Xi}(\flat,\wp;\delta)-\sum_{p=0}^{q}\tilde{\Xi}_p(\flat,\wp;\delta)|| \\
&=||\sum_{p=0}^{\infty}\tilde{\Xi}_p(\flat,\wp;\delta)|| \\
&\le\sum_{p=q+1}^{\infty}||\tilde{\Xi}_p(\flat,\wp;\delta)|| \\
&\le\sum_{p=q+1}^{\infty}\lambda^q||\tilde{\Xi}_0(\flat,\wp;\delta)|| \\
&\lambda^{q+1}(1+\lambda+\lambda^2+\cdots)||\tilde{\Xi}_0(\flat,\wp;\delta)|| \\
&\le\frac{\lambda^{q+1}}{1-\lambda}||\tilde{\Xi}_0(\flat,\wp;\delta)||
\end{aligned} \quad 28$$

hence prove.

5.0 Numerical Example:

Example 1 *Consider the fuzzy fractional Klein-Gordan equation as*

$$\frac{\partial^\alpha\tilde{\Xi}(\flat,\wp;\delta)}{\partial\wp^\alpha}-\frac{\partial^2\tilde{\Xi}(\flat,\wp;\delta)}{\partial\flat^2}+\tilde{\Xi}(\flat,\wp;\delta)=0, \quad 1<\alpha\le2, \quad 29$$

With the fuzzy initial conditions

$$\tilde{\Xi}(\flat,0;\delta)=0 \quad \text{and} \quad \tilde{\Xi}_{\wp}(\flat,0;\delta)=\tilde{k}(\delta)\odot\flat \quad 30$$

where $\tilde{k}(\delta)=[\delta-1,1-\delta]$.

The outcome of the above scheme of Eq. 22 and Eq. 23, is

$$\underline{\Xi}_0(\flat,\wp;\delta)=\flat(\delta-1)\wp$$

$$\overline{\Xi}_0(\flat,\wp;\delta)=\flat(1-\delta)\wp$$

$$\underline{\Xi}_1(\flat,\wp;\delta)=-\flat(\delta-1)\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\overline{\Xi}_1(\flat,\wp;\delta)=-\flat(1-\delta)\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\underline{\Xi}_2(\flat,\wp;\delta)=\flat(\delta-1)\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

ISSN 2063-5346 *Section H-Research paper*

$$\overline{\Xi}_2(\flat,\wp;\delta) = \flat(1-\delta)\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\underline{\Xi}_3(\flat,\wp;\delta) = -\flat(\delta-1)\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\overline{\Xi}_3(\flat,\wp;\delta) = -\flat(1-\delta)\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\underline{\Xi}_4(\flat,\wp;\delta) = \flat(\delta-1)\frac{\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}$$

$$\overline{\Xi}_4(\flat,\wp;\delta) = \flat(1-\delta)\frac{\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}$$

$$\underline{\Xi}_5(\flat,\wp;\delta) = -\flat(\delta-1)\frac{\wp^{5\alpha+1}}{\Gamma(5\alpha+2)}$$

$$\overline{\Xi}_4(\flat,\wp;\delta) = -\flat(1-\delta)\frac{\wp^{5\alpha+1}}{\Gamma(5\alpha+2)}$$

$$\vdots$$

and so on.

Thus the lower and upper forms of a solution are given as

$$\underline{\Xi}(\flat,\wp;\delta) = \underline{\Xi}_0(\flat,\wp;\delta) + \underline{\Xi}_1(\flat,\wp;\delta) + \underline{\Xi}_2(\flat,\wp;\delta) + \underline{\Xi}_3(\flat,\wp;\delta) + \cdots$$

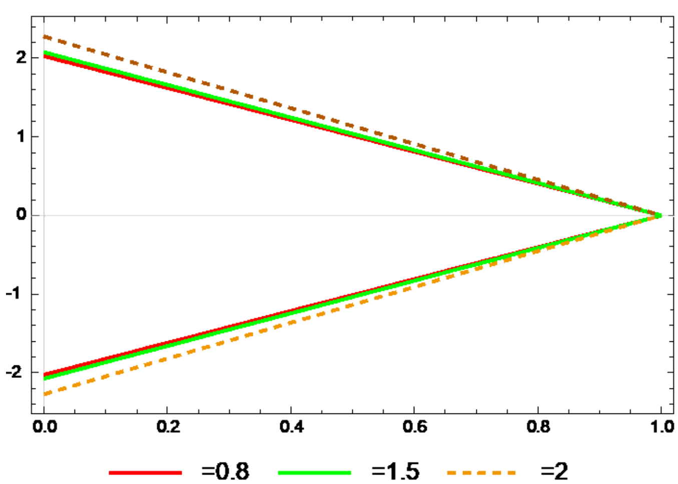

Fig. 2: 2-Dimensional simulation of Example (1) at α=0.8,1.5,2 and uncertainty δ∈[0,1]

$$\overline{\Xi}(\flat,\wp;\delta) = \overline{\Xi}_0(\flat,\wp;\delta) + \overline{\Xi}_1(\flat,\wp;\delta) + \overline{\Xi}_2(\flat,\wp;\delta) + \overline{\Xi}_3(\flat,\wp;\delta) + \cdots$$

$$\underline{\Xi}(\flat,\wp;\delta) = \flat(1-\delta)\wp\begin{bmatrix}1-\frac{\wp^{\alpha}}{\Gamma(\alpha+2)}+\frac{\wp^{2\alpha}}{\Gamma(2\alpha+2)}-\frac{\wp^{3\alpha}}{\Gamma(3\alpha+2)}\\ +\frac{\wp^{4\alpha}}{\Gamma(4\alpha+2)}-\frac{\wp^{5\alpha}}{\Gamma(5\alpha+2)}+\cdots\end{bmatrix}$$

$$\overline{\Xi}(\flat,\wp;\delta) = \flat(1-\delta)\wp\begin{bmatrix}1-\frac{\wp^{\alpha}}{\Gamma(\alpha+2)}+\frac{\wp^{2\alpha}}{\Gamma(2\alpha+2)}-\frac{\wp^{3\alpha}}{\Gamma(3\alpha+2)}\\ +\frac{\wp^{4\alpha}}{\Gamma(4\alpha+2)}-\frac{\wp^{5\alpha}}{\Gamma(5\alpha+2)}+\cdots\end{bmatrix} \quad 31$$

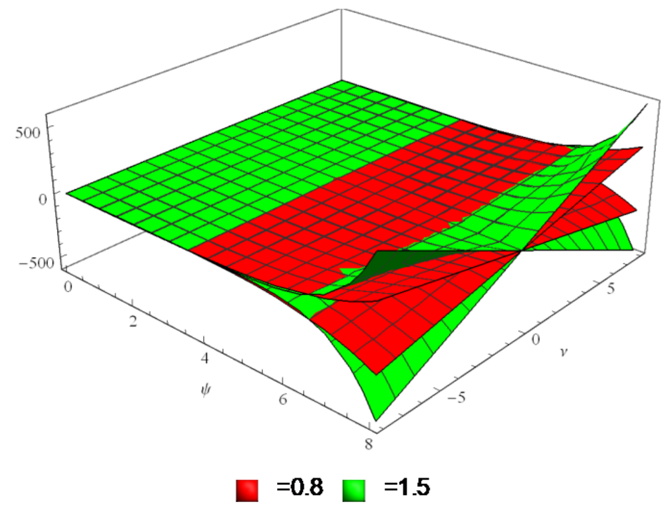

Fig. 1:: 3-Dimensional simulation of Example (1) at α=0.8,1.5 and uncertainty δ∈[0,1]

If put $\alpha = 2$ in equation (31) then obtain

$$\tilde{\Xi}(\flat,\wp;\delta) = \tilde{k}(\delta)\,\flat\,\sin(\wp)$$

which is the exact solution.

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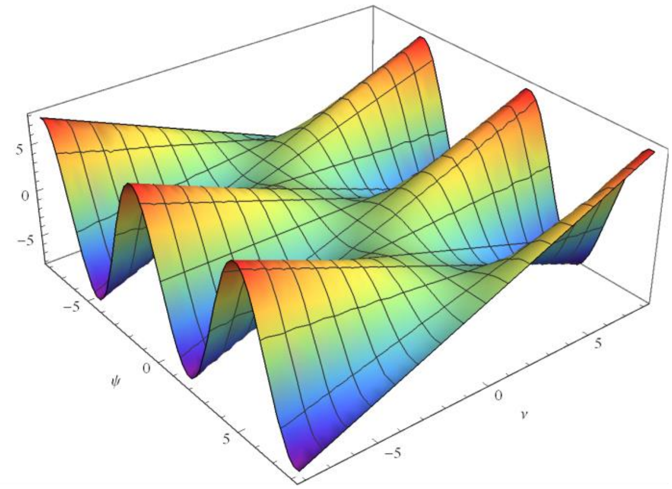

Fig. 3: Exact Geometrical representation of Eq. 31 at α=2

Example 2 *Consider the non-homogeneous fuzzy fractional Klein-Gordan equation as*

$$\frac{\partial^{\alpha}\tilde{\Xi}(\flat,\wp;\delta)}{\partial\wp^{\alpha}}-\frac{\partial^{2}\tilde{\Xi}(\flat,\wp;\delta)}{\partial\flat^{2}}+\tilde{\Xi}(\flat,\wp;\delta)=2\sin(\flat),\quad 1<\alpha\leq 2, \qquad 32$$

With the fuzzy initial conditions

$$\tilde{\Xi}(\flat,0;\delta)=\tilde{k}(\delta)\sin(\flat)\quad\text{and}\quad\tilde{\Xi}_{\wp}(\flat,0;\delta)=\tilde{k}(\delta) \qquad 33$$

where $\tilde{k}(\delta)=[\delta-1,1-\delta]$.

The outcome of the above scheme of Eq. 22 and Eq. 23, is

$$\underline{\Xi}_0(\flat,\wp;\delta)=\frac{2\sin(\flat)\wp^{\alpha}}{\Gamma(\alpha+1)}-\sin(\flat)+\sin(\flat)\delta+\wp\delta-\wp$$

$$\overline{\Xi}_0(\flat,\wp;\delta)=\frac{2\sin(\flat)\wp^{\alpha}}{\Gamma(\alpha+1)}+\sin(\flat)-\sin(\flat)\delta-\wp\delta+\wp$$

$$\underline{\Xi}_1(\flat,\wp;\delta)=-\frac{4\sin(\flat)\wp^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}+\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{2\sin(\flat)\delta\wp^{\alpha}}{\Gamma(\alpha+1)}-\frac{\delta\wp^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\overline{\Xi}_1(\flat,\wp;\delta)=-\frac{4\sin(\flat)\wp^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}-\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{2\sin(\flat)\delta\wp^{\alpha}}{\Gamma(\alpha+1)}+\frac{\delta\wp^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\underline{\Xi}_2(\flat,\wp;\delta)=-\frac{4\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{4\delta\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{\delta\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

ISSN 2063-5346

$$\overline{\Xi}_2(\flat,\wp;\delta)=\frac{4\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}+\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{4\delta\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}-\frac{r\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\underline{\Xi}_3(\flat,\wp;\delta)=\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{16\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}+\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}-\frac{8\delta\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{\delta\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\overline{\Xi}_3(\flat,\wp;\delta)=-\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{16\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}-\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{8\delta\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}+\frac{\delta\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\underline{\Xi}_4(\flat,\wp;\delta)=\left\{\begin{matrix}-\frac{4\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{32\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}\\+\frac{2\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}-\frac{\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}\\+\frac{16\delta\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}+\frac{\delta\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}\end{matrix}\right\}$$

$$\overline{\Xi}_4(\flat,\wp;\delta)=\left\{\begin{matrix}-\frac{4\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{32\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}\\+\frac{2\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}-\frac{\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}\\+\frac{16\delta\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}+\frac{\delta\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}\end{matrix}\right\}$$

$$\vdots$$

and so on.

Thus the lower and upper forms of a solution are given as

$$\underline{\Xi}(\flat,\wp;\delta)=\underline{\Xi}_0(\flat,\wp;\delta)+\underline{\Xi}_1(\flat,\wp;\delta)+\underline{\Xi}_2(\flat,\wp;\delta)+\underline{\Xi}_3(\flat,\wp;\delta)+\cdots$$

$$\overline{\Xi}(\flat,\wp;\delta)=\overline{\Xi}_0(\flat,\wp;\delta)+\overline{\Xi}_1(\flat,\wp;\delta)+\overline{\Xi}_2(\flat,\wp;\delta)+\overline{\Xi}_3(\flat,\wp;\delta)+\cdots$$

$$\underline{\Xi}(\flat,\wp;\delta)=\left\{\begin{matrix}-\frac{12\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{24\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{48\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}+\frac{6\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}+\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}\\+\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}-\frac{\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}-\sin(\flat)+\frac{4r\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}-\frac{8r\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}+\frac{16r\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}\\-\frac{2r\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}-\frac{r\wp^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{r\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{r\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{r\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}+r\sin(\flat)+r\wp-\wp\end{matrix}\right\}$$

$$\overline{\Xi}(\flat,\wp;\delta)=\left\{\begin{matrix}-\frac{4\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{8\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}-\frac{48\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}+\frac{2\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}-\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}\\-\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}-\frac{\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}+\sin(\flat)-\frac{4r\wp^{2\alpha}\sin(\flat)}{\Gamma(2\alpha+1)}+\frac{8r\wp^{3\alpha}\sin(\flat)}{\Gamma(3\alpha+1)}+\frac{16r\wp^{4\alpha}\sin(\flat)}{\Gamma(4\alpha+1)}\\+\frac{2r\wp^{\alpha}\sin(\flat)}{\Gamma(\alpha+1)}+\frac{r\wp^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{r\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{r\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{r\wp^{4\alpha+1}}{\Gamma(4\alpha+2)}-r\sin(\flat)-r\wp+\wp\end{matrix}\right\}$$

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If put $\alpha = 2$ in Eq. 34 then obtain

$$\tilde{\Xi}(\flat, \wp; \delta) = \tilde{k}(\delta)(\sin(\flat) + \sin(\wp))$$

which is exact solution.

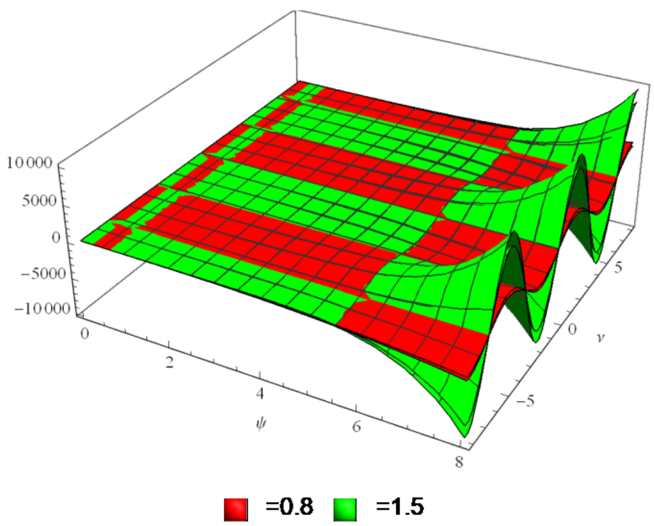

Fig. 4: 3-Dimensional simulation of (2) at α=0.8,1.5 and uncertainty δ∈[0,1]

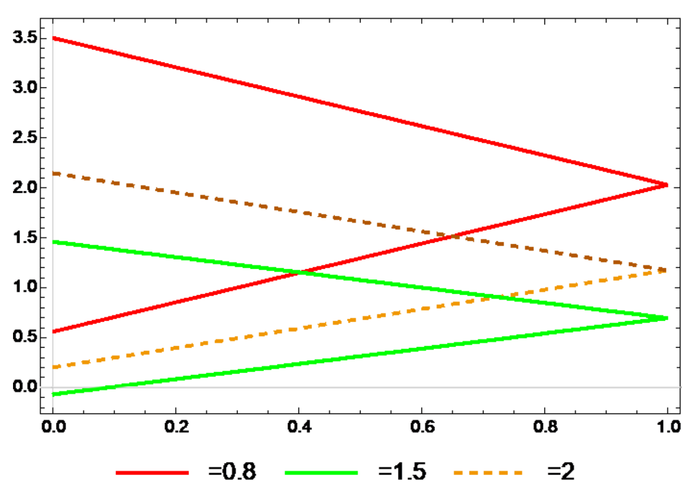

Fig.5: 2-Dimensional simulation of (2) at α=0.8,1.5,2 and uncertainty δ∈[0,1]

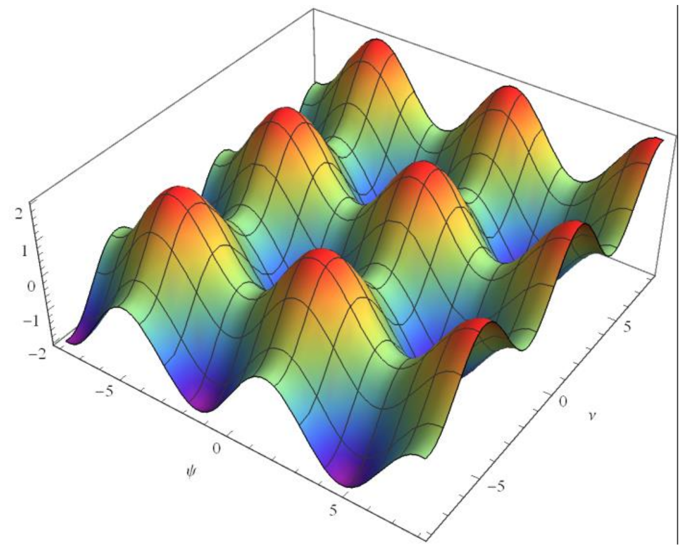

Fig. 6: Exact Geometrical representation of Eq. 31 at α=2

Example 3 *Consider the fuzzy fractional Klein-Gordan equation as*

$$\frac{\partial^\alpha \tilde{\Xi}(\flat, \wp; \delta)}{\partial \wp^\alpha} - \frac{\partial^2 \tilde{\Xi}(\flat, \wp; \delta)}{\partial \flat^2} - \left(\frac{\partial \tilde{\Xi}(\flat, \wp; \delta)}{\partial \flat}\right)^2 - \tilde{\Xi}^2(\flat, \wp; \delta) = 0, \quad 0 < \alpha \leq 1, \qquad 35$$

With the fuzzy initial condition

$$\tilde{\Xi}(\flat, 0; \delta) = 0 \quad \text{and} \quad \tilde{\Xi}_{\wp}(\flat, 0; \delta) = \tilde{k}(\delta) e^{\flat} \qquad 36$$

where $\tilde{k}(\delta) = [\delta - 1, 1 - \delta]$.

The outcome of the above scheme of Eq. 22 and Eq. 23, is

$$\underline{\Xi}_0(\flat, \wp; \delta) = \wp(\delta - 1) e^{\flat}$$

$$\overline{\Xi}_0(\flat, \wp; \delta) = \wp(1 - \delta) e^{\flat}$$

$$\underline{\Xi}_1(\flat,\wp;\delta) = e^{\flat}(\delta-1)\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\overline{\Xi}_1(\flat,\wp;\delta) = e^{\flat}(1-\delta)\frac{\wp^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$\underline{\Xi}_2(\flat,\wp;\delta) = e^{\flat}(\delta-1)\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\overline{\Xi}_2(\flat,\wp;\delta) = e^{\flat}(1-\delta)\frac{\wp^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$\underline{\Xi}_3(\flat,\wp;\delta) = e^{\flat}(\delta-1)\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\overline{\Xi}_3(\flat,\wp;\delta) = e^{\flat}(1-\delta)\frac{\wp^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\vdots$$

and so on.

Thus the lower and upper forms of a solution are given as

$$\underline{\Xi}(\flat,\wp;\delta) = e^{\flat}(\delta-1)\wp\left(1+\frac{\wp^{\alpha}}{\Gamma(\alpha+2)}+\frac{\wp^{2\alpha}}{\Gamma(2\alpha+2)}+\frac{\wp^{3\alpha}}{\Gamma(3\alpha+2)}+\frac{\wp^{4\alpha}}{\Gamma(4\alpha+2)}+\cdots\right)$$

$$\overline{\Xi}(\flat,\wp;\delta) = e^{\flat}(1-\delta)\wp\left(1+\frac{\wp^{\alpha}}{\Gamma(\alpha+2)}+\frac{\wp^{2\alpha}}{\Gamma(2\alpha+2)}+\frac{\wp^{3\alpha}}{\Gamma(3\alpha+2)}+\frac{\wp^{4\alpha}}{\Gamma(4\alpha+2)}+\cdots\right) \quad 37$$

If put $\alpha = 2$ in Eq. 37 then obtain

$$\tilde{\Xi}(\flat,\wp;\delta) = \tilde{k}(\delta)(e^{\flat}\sinh(\wp))$$

which is the exact solution.

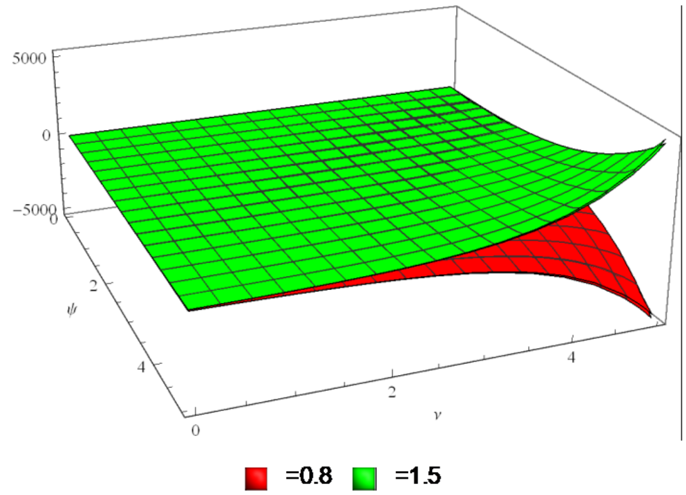

Fig. 7: 3-Dimensional simulation of Example (3) at α=0.6,0.8 and uncertainty δ ∈[0,1]

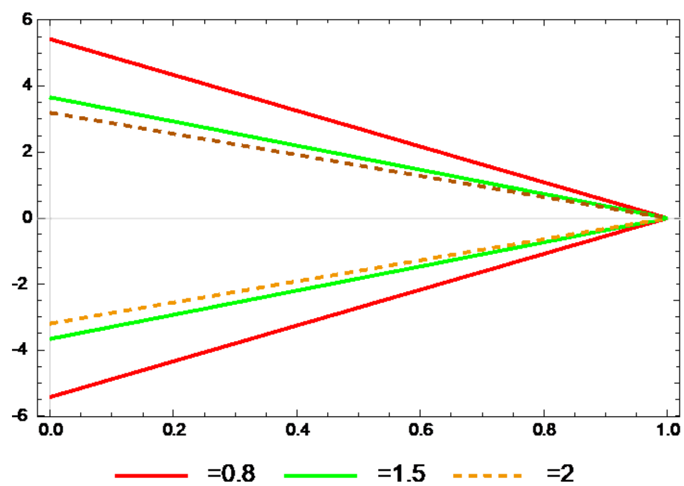

Fig.8: 2-Dimensional simulation of Example (3) at α=0.6,0.8,1 and uncertainty δ∈[0,1]

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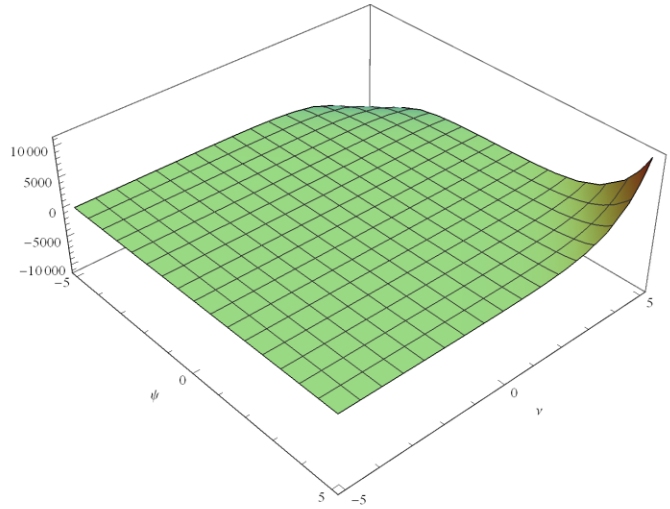

Fig. 9: Exact Geometrical representation of Eq. 31 at α=2

Table 1: Error Analysis of Example 1

	$\alpha = 2$					
	Lower			**Upper**		
r	**Approximate**	**Exact**	**Error**	**Approximate**	**Exact**	**Error**
0	-1.81859	-1.81859	2.58173×10^{-6}	1.81859	1.81859	2.58173×10^{-6}
0.2	-1.45487	-1.45488	2.06538×10^{-6}	1.45487	1.45488	2.06538×10^{-6}
0.4	-1.09116	-1.09116	1.54904×10^{-6}	1.09116	1.09116	1.54904×10^{-6}
0.6	-0.727437	-0.727438	1.03269×10^{-6}	7.27437×10^{-1}	7.27438×10^{-1}	1.03269×10^{-6}
0.8	-0.363718	-0.363719	5.16345×10^{-7}	3.63718×10^{-1}	3.63719×10^{-1}	5.16345×10^{-7}
1	0	0	0.00000	0.00000	0.00000	0.00000

Table 2: Error Analysis of Example 2

	$\alpha = 2$					
	Lower			**Upper**		
r	**Approximate**	**Exact**	**Error**	**Approximate**	**Exact**	**Error**
0	3.4699×10^{-1}	-1.81859	2.16559	4.5101×10^{-1}	1.81859	1.36758
0.2	3.57392×10^{-1}	-1.45488	1.81227	4.40608×10^{-1}	1.45488	1.01427
0.4	3.67794×10^{-1}	-1.09116	1.45895	4.30206×10^{-1}	1.09116	6.60951×10^{-1}
0.6	3.78196×10^{-1}	-0.727438	1.10563	4.19804×10^{-1}	7.27438×10^{-1}	3.07634×10^{-1}
0.8	3.88598×10^{-1}	-0.363719	7.52317×10^{-1}	4.09402×10^{-1}	3.63719×10^{-1}	4.56834×10^{-2}
1	3.99×10^{-1}	0	3.99×10^{-1}	3.99×10^{-1}	0	3.99×10^{-1}

Table 3: Error Analysis of Example 3

$\alpha = 2$						
	Lower			Upper		
r	**Approximate**	**Exact**	**Error**	**Approximate**	**Exact**	**Error**
0	-26.7987	-26.7991	3.89017×10^{-4}	2.67987×10^{1}	2.67991×10^{1}	3.89017×10^{-4}
0.2	-21.4389	-21.4393	3.11213×10^{-4}	2.14389×10^{1}	2.14393×10^{1}	3.11213×10^{-4}
0.4	-16.0792	-16.0794	2.3341×10^{-4}	1.60792×10^{1}	1.60794×10^{1}	2.3341×10^{-4}
0.6	-10.7195	-10.7196	1.55607×10^{-4}	1.07195×10^{1}	1.07196×10^{1}	1.55607×10^{-4}
0.8	-5.35974	-5.35982	7.78034×10^{-5}	5.35974	5.35982	7.78034×10^{-5}
1	0	0	0	0	0	0

Conclusion:

In this paper, fuzzy time-fractional telegraph equations were successfully solved using the fuzzy Laplace-Adomian decomposition method. The FLADM has been shown to be a powerful tool for solving nonlinear partial differential equations with practical applications, and our results demonstrate its effectiveness for solving fuzzy fractional Klein-Gordan. The use of the Wolfram Mathematica 11.3 program for computations adds to the reliability and accuracy of the results. Overall, this study provides valuable insights into solving fuzzy time fractional telegraph equations, and it presents new opportunities for further research in this field.

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Authors' declaration:

- Conflicts of Interest: None.

Authors vouch for the authenticity of all the manuscript's drawings and tables.

Authors' Contributions Statement:

K. K. proposed the Concepts, ideas, method, analysis, and Drafting of the manuscript. V. N. designed the manuscript. S. G. read the manuscript and revised it. S. T. interpreted and plotted the graphs of the solution of examples using Mathematica 11.3. All authors read and approved the final manuscript.

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