A Non-Dyadic Haar wavelet approach to Numerical solution of non-linear Klein-Gordon Equation.

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Abstract:

In the research paper, Klein Gordon Equation which is a non-linear partial differential equation of order two is discussed by using HS-3W method. A variety of physical phenomena are discussed in the field of Engineering and sciences by using this equation. Space and time approximation is done by using the proposed technique and converting the equation into system of algebraic equations and the non-linearities present in the equation is solved by using Quasilinearisation approach. To check the applicability of the method, proposed scheme is applied on various examples which shows the methods accuracy and compatibility with good results. By using MATLAB, graphical representation of exact, approximate solution and absolute error is shown.

Keywords: Nonlinear Klein Gordon equation (KGE), Partial differential equation, Haar scale-3 wavelet method (HS3WM), Quasilinearisation technique.

1. Introduction

In various fields such as plasma physics, solid state physics, fluid physics, chemical kinetics and mathematical biology non-linear phenomena plays a vital role for discussing the solution of the physical problems developed in these fields which are represented in terms of partial differential equations. To handle with these kinds of difficulties different numerical and analytical method are used for finding the solution of these equations. As per the literature review, it observed that wavelet-based methods are one of the compatible tools for searching the solution of these kinds of real-life problems. As it is enough sufficient to tackle with these kinds of complexities. Several articles have been already published to discuss the applicability of wavelet method for evaluating the solution non-linear and higher order complex differential equations. Haar wavelet is simplest wavelet of the wavelet family with a simple structure and it compact as well as orthonormal. In the present work, a nonlinear KGE is discussed using HS-3W method. It has application in the sector of applied physics like as phenomena of field theory and quantum mechanics. Basically, KGE belongs to family of wave equations used to study the behaviour of particle in motions at high velocities with high energy.
Nonlinear KGE expressed as:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} + \mu\, u - \epsilon u^2 = m(z,t), \qquad (z,t) \in [0,1] \times [0,T] \tag{1}$$

Having initial constraints as:

$$u(z,0) = \eta_1(z) \quad (2)$$

$$\frac{\partial u}{\partial t}(z,0) = \eta_2(z) \quad (3)$$

and the boundary constraints

$$u(0,t) = \gamma_1(t), \quad t \in [0,T] \quad (4)$$

$$u(1,t) = \gamma_2(t), \quad t \in [0,T] \quad (5)$$

Where μ, ϵ, m are known constants, $\eta_1(z), \eta_2(z), \gamma_1(t)$, $\gamma_2(t)$, $m(z,t)$ are the given functions and here value of $u(z,t)$ is to be determined.

Different researchers, put a lot of efforts to discuss the structure of this equation by using multiple numerical methods such as Radial Basis function is used to discuss one dimensional non-linear KGE with quadratic and cubic non-linearity [1], Adomain decomposition [5-6],Decomposition method [3-4], VIM [9-10], HPM[8], Auxiliary equation method [2], Polynomial wavelets [7], Legendre wavelet [13-14], Hermite wavelet operational matrix [11], Laguerre wavelet [12], Chebyshev Wavelet Method (CWM) [15], Haar wavelet collocation method [16].

As per the observation, non-linear KGE is not yet discussed by Haar scale 3 wavelet method in the previous work that motivated us to generate a method to solve this kind of equation and results of HS-3WM methods are more accurate as compare to HS-2WM.

2. Structure of HS-3W with their integrals:

The expressions for HS-3W family [18], [20], [28] in mathematical way is represented as:

Haar scaling function

$$h_i(m) = \varphi(m) = \begin{cases} 1 & a \le m < b \\ 0 & elsewhere \end{cases} \quad (6)$$

$$for \quad i = 1$$

Haar Symmetric Wavelet Function (7)

$$h_i(m) = \psi^1(3^j m - k) = \frac{1}{\sqrt{2}} \begin{cases} -1 & a_{11}(i) \le m < a_{12}(i) \\ 2 & a_{12}(i) \le m < a_{13}(i) \\ -1 & a_{13}(i) \le m < a_{14}(i) \\ 0 & elsewhere \end{cases},$$

$$for\ i = 2,4, \dots 3p - 1$$

(8)

Haar Anti-Symmetric Wavelet Function

$$h_i(m) = \psi^2(3^j m - k) = \sqrt{\frac{3}{2}}\begin{cases} 1 & a_{11}(i) \le m < a_{12}(i) \\ 0 & a_{12}(i) \le m < a_{13}(i) \\ -1 & a_{13}(i) \le m < a_{14}(i) \\ 0 & elsewhere \end{cases},$$

$$for \quad i = 3,6, \ldots 3p$$

$a_{11}(i) = (b-a)\frac{k}{p}$, $a_{12}(i) = (b-a)\,\frac{3k+1}{3p}$,
$a_{13}(i) = (b-a)\frac{(3k+2)}{3p}$, $a_{14}(i) = (b-a)\frac{k+1}{p}$,
Here $p = 3^j$, $k = 0,1,2, \ldots, p-1$, $j = 0,1,2, \ldots$.
Now, over the interval [A, B) one can easily integrate the equations by using Riemann Liouville Integral formula for desired number of times as defined below

$$\varphi_{i,s}(q) = \int_0^z \varphi_{1,s-1}(q)dq = \begin{cases} \frac{q^s}{\Gamma(s+1)}; q \in [a,b) \\ 0; otherwise \end{cases} \quad for\ i = 1$$

$$\varphi_{i,s}^1(q) = \int_0^z \varphi_{i,s-1}^1(q)\,dq =$$

$$\frac{1}{\sqrt{2}}\begin{cases} 0 & for \quad q \in [0, a_{11}(i)) \\ \frac{-1}{\Gamma(s+1)}\left(q - a_{11}(i)\right)^s & for \quad q \in [a_{11}(i), a_{12}(i)) \\ \frac{1}{\Gamma(s+1)}[-\left(q - a_{11}(i)\right)^s + 3\left(q - a_{12}(i)\right)^s] & for \quad q \in [a_{12}(i), a_{13}(i)) \\ \frac{1}{\Gamma(s+1)}[-\left(q - a_{11}(i)\right)^s + 3\left(q - a_{12}(i)\right)^s - 3\left(q - a_{13}(i)\right)^s] & for\ q \in [a_{13}(i), a_{14}(i)) \\ \frac{1}{\Gamma(s+1)}[-\left(q - a_{11}(i)\right)^s + 3\left(q - a_{12}(i)\right)^s - 3\left(q - a_{13}(i)\right)^s + \left(q - a_{14}(i)\right)^s] & for \quad q \in [a_{14}(i), 1) \end{cases}$$

$$for \quad i = 2,4,6,8, \cdots, 3p-1$$

$$\varphi_{i,s}^2(q) = \int_0^z \varphi_{i,s-1}^2(q)\,dq =$$

$$\sqrt{\frac{3}{2}}\begin{cases} 0 & for\ q \in [0, a_{11}(i)) \\ \frac{1}{\Gamma(s+1)}\left(q - a_{11}(i)\right)^s & for\ q \in [a_{11}(i), a_{12}(i)) \\ \frac{1}{\Gamma(s+1)}[\left(q - a_{11}(i)\right)^s - \left(q - a_{12}(i)\right)^s] & for \quad q \in [a_{12}(i), a_{13}(i)) \\ \frac{1}{\Gamma(s+1)}[\left(q - a_{11}(i)\right)^s - \left(q - a_{12}(i)\right)^s - \left(q - a_{13}(i)\right)^s] & for\ q \in [a_{13}(i), a_{14}(i)) \\ \frac{1}{\Gamma(s+1)}[\left(q - a_{11}(i)\right)^s - \left(q - a_{12}(i)\right)^s - \left(q - a_{13}(i)\right)^s + \left(q - a_{14}(i)\right)^s] & for\ q \in [a_{14}(i), 1) \end{cases}$$

$$for \quad i = 3,5,7,9, \cdots, 3p$$

3. Approximation of solution

If $u(z,t) = L_2(R)$, using Haar Scale 3 wavelets it can be approximated numerically as

$$u(z,t) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{il}\, h_i(z) h_n(t)$$

By using the method $a_{in}'s$ will be evaluated. But a truncated series up to $3p \times 3p$ terms can be considered for computational purpose. By considering the $3p \times 3p$ terms, we get

$$u(z,t) \approx u_{3p}(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{il}\, h_i(z) h_n(t)$$

where $p = 3^j,\ j = 0,1,2, \ldots$

4. Quasi-linearisation process

For linearizing nonlinear differential equations, quasi-linearization strategy is used which is a generalized form of the Newton Raphson method. Quadratically, it converges to the exact value. If we have a non-linear term, we must employ the recurrence relation shown below:

$$\mu^2 = f(\mu^2) = f(x, \mu^2)$$

$$[\mu^2]_{s+1} = [\mu^2]_s + [\mu_{s+1} - \mu_s]\left(\frac{\partial}{\partial u}(\mu^2)\right)_s$$

$$(\mu^2)_{s+1} = (\mu^2)_s + (\mu_{s+1} - \mu_s)\, 2\mu_s$$

$$(\mu^2)_{s+1} = (\mu^2)_s + 2\mu_{s+1}(\mu)_s - 2(\mu^2)_s$$

$$(\mu^2)_{s+1} = 2(\mu)_{s+1}\mu_s - (\mu^2)_s$$

5. Method of Solution

Space and time variables containing Higher order derivatives are approximated with the help of HS-3W' as explained below:

$$u_{zztt}(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\, h_i(z)\, h_n(t) \tag{9}$$

To perform integration with respect to z, the lower limit is set to zero, while the upper limit is z, the above equation converted in to

$$u_{ztt}(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\, P_{1,i}(z)\, h_n(t) + \varphi_{ztt}(0,t) \tag{10}$$

Again, integration with respect to z, the lower limit is set to zero, while the upper limit is, the value of $u_{ztt}(0,t)$ is given by

$$u_{ztt}(0,t) = \big(u_{tt}(1,t) - u_{tt}(0,t)\big) - \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\, P_{2,i}(1)\, h_n(t) \tag{11}$$

and equation becomes,

$$u_{ztt}(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left(P_{1,i}(z) - P_{2,i}(1)\right) h_n(t) + \left(u_{tt}(1,t) - u_{tt}(0,t)\right) \tag{12}$$

Again, integration with respect to z, the lower limit is set to zero, while the upper limit is z

$$u_{tt}(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) h_n(t) + z\,u_{tt}(1,t) + (1 - z)\,u_{tt}(0,t) \tag{13}$$

between the limit 0 $to\ t$, integrating w.r.t t and after that on applying the boundary conditions, we get

$$u_t(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) P_{1,n}(t) + z\,(u_t(1,t) - \psi_2(1)) + (1 - z)\left(u_t(0,t) - \psi_2(0)\right) + \psi_2(z)$$

Integrate w.r.t t in between the given limit implies

$$u(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) P_{2,n}(t) + z\,(\xi_2(t) - \xi_2(0)) - z\,t\,\psi_2(1) + (1 - z)\left(\xi_1(t) - \xi_1(0)\right) - (1 - z)\,t\psi_2(0) + t\psi_2(z) \tag{14}$$

On Differentiating the above equation with respect z two times,

$$u_{zz}(z,t) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\,h_i(z)P_{2,n}(t) + t(\psi_2)_{zz}(z) \tag{15}$$

$$\sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) h_n(t) + z\left(\xi_2(t)\right)_{tt} + (1 - z)\left(\xi_1(t)\right)_{tt} + \alpha\left(\sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) P_{2,n}(t) + z(\xi_2(t) - \xi_2(0)) - zt\,\psi_2(1) + (1 - z)\left(\xi_1(t) - \xi_1(0)\right) - (1 - z)\,t\psi_2(0) + t\psi_2(z)\right) = \sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\,h_i(z)P_{2,n}(t) + t(\psi_2)_{zz}(z) + m(z,t)$$

$$\sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\left[\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) h_n(t) + \alpha\left(\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) P_{2,n}(t)\right) - h_i(z)P_{2,n}(t)\right] =$$
$$m(z,t) + t(\psi_2)_{zz}(z) - \left(z\left(\xi_2(t)\right)_{tt}, \ + (1 - z)\left(\xi_1(t)\right)_{tt}\right) - \alpha\left(z\,(\xi_2(t) - \xi_2(0)) - z\,t\,\psi_2(1) + (1 - z)\left(\xi_1(t) - \xi_1(0)\right) - (1 - z)\,t\psi_2(0) + t\psi_2(z)\right)$$

Now discretizing the variable as $z \to z_r$, $t \to t_s$ where $x_r = \frac{2r-1}{6p}$, $t_s = \frac{2s-1}{6p}$, $r, s = 1,2, \dots \dots 3p$ in the above equations we get the following system of algebraic equations

$$\sum_{i=1}^{3p}\sum_{n=1}^{3p} a_{in}\,R_{i,n,r,s} = F(r,s) \tag{16}$$

Where $R_{i,n,r,s} = \left[\left(P_{2,i}(z_r) - z_r\,P_{2,i}(1)\right) h_n(t_s) + \alpha\left(\left(P_{2,i}(z_r) - z_r\,P_{2,i}(1)\right) P_{2,n}(t_s)\right) - h_i(z_r)P_{2,n}(t_s)\right]$

$$F(r,s) = m(z_r, t_s) + t(\psi_2)_{zz}(z_r) - \left(z_r\left(\xi_2(t_s)\right)_{tt}, \quad + (1-z_r)\left(\xi_1(t_s)\right)_{tt}\right) - \alpha\left(z_r\,(\xi_2(t_s) - \xi_2(0)) - z_r\, t_s\, \psi_2(1) + (1-z_r)\left(\xi_1(t_s) - \xi_1(0)\right) - (1-z_r)\, t\psi_2(0) + t_s\psi_2(z_r)\right) \quad (17)$$

The system mentioned above is simplified into a system of algebraic equations, which is then reduced to the following set of 4D-arrays

$$A_{3p\times 3p} R_{3p\times 3p\times 3p\times 3p} = F_{3p\times 3p}$$

Using the given transformations, the aforementioned system of arrays is transformed and reduced to the following matrix system.

$$a_{i\ell} = b_\lambda \ \ and\ F_{rs} = G_\mu$$

$$B_{1\times(3p)^2} S_{(3p)^2\times(3p)^2} = G_{1\times(3p)^2}$$

Where $\lambda = 3p(i-1) + l$ and $\mu = 3p(r-1) + s$

The values of b_λ can be calculated for various n values (n=1, 2, ...) using MATLAB program and the Thomas algorithm to solve the system of equations mentioned above. By applying the transformation mentioned earlier, the original wavelet coefficients $a_{i\ell}$ can be retrieved. To obtain the final solution of the problem, these coefficients will be utilized in the equations for different t_n values (n=0, 1, 2, ...).

6. Error analysis

To assess the accuracy of the proposed technique for non-linear KGE, it is applied at different levels of resolutions to examine its compatibility. The present method is used to calculate absolute errors, L_∞error, and L_2error for various problems using the formulas discussed above.:

$$\text{Absolute error} = |u_{exact}(z_r, t_s) - u_{num}(z_r, t_s)| \quad (18)$$

$$L_\infty = \max_{r,s} |u_{exact}(z_r, t_s) - u_{num}(z_r, t_s)| \quad (19)$$

$$L_2 = \frac{\sqrt{\sum_{l=1}^{3p} |u_{exact}(z_r,t_s) - u_{num}(z_r,t_s)|^2}}{\sqrt{\sum_{l=1}^{3p} |u_{exact}(z_r,t_s)|^2}} \quad (20)$$

<u>Numerical experiment. 1</u>: - "A *Non-linear KGE*":

$$\frac{\partial^2 u}{\partial t^2} - \omega \frac{\partial^2 u}{\partial z^2} + \mu\, u - \epsilon u^2 = m(z,t), (z,t) \in [0,1] \times [0,T], \quad (21)$$

With initial conditions,

$$u(z,0) = 0 \quad z \in [0,1], \ \frac{\partial u}{\partial t}(z,0) = 0, z \in [0,1]$$

and the boundary constraints

$$u(0,t) = 0 \quad , \ u(1,t) = t^3 \quad t \in [0,T]$$

with $\omega = -1, \mu = 0, \epsilon = 1 \;\&\; m(z,t) = 6zt(z^2 - t^2) + z^6 t^6$ (22)

Exact solution for the problem: $u(x,t) = z^3 t^3$ (23)

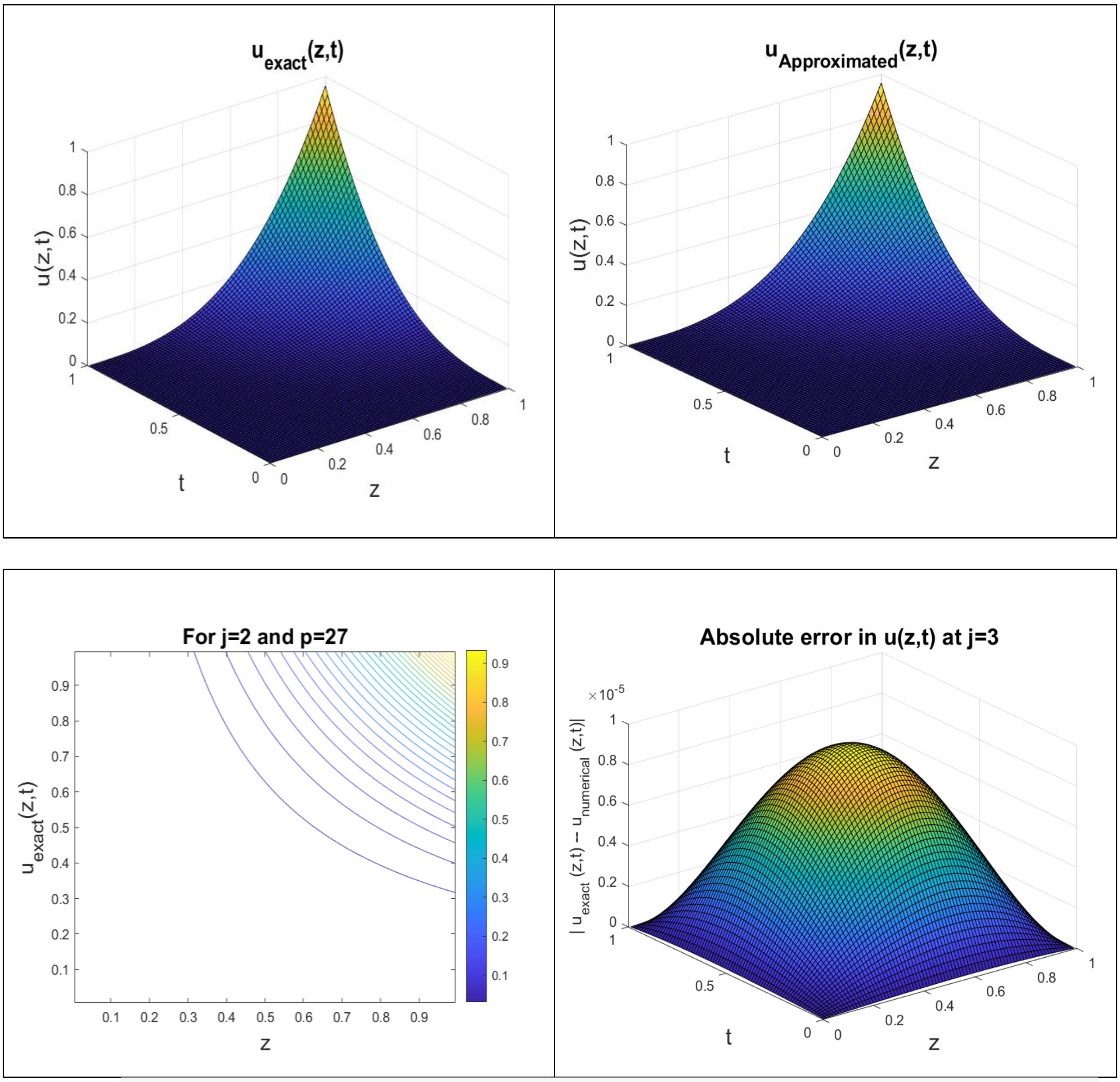


Figure 1: The solution for numerical Experiment no. 1 includes four graphical representations: exact solution, approximate solution, contour view of the exact solution, and absolute error.

Using current scheme, proposed the numerical solution in the following form

$$u(z,t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{i\ell}\left(P_{2,i}(z) - z\,P_{2,i}(1)\right) Q_{2,\ell}(t) - z\,t\,\psi_2(1) + z\,(\xi_2(t) - \xi_2(0)) + (1-z)\big(\xi_1(t) - \xi_1(0)\big) - (1-z)\,t\psi_2(0) + t\psi_2(z) \quad (24)$$

Table 1: Table 3: At different of $'j'$, value L_2 and L_∞ errors for Test problem 1

Resolution Level	$J = 1$	$J = 2$	$J = 3$	
L_2-error	2.69334632e-03	2.9622179667e-04	3.28768376e-05	2.5e-04 [25]
L_∞-error	7.62593758e-04	8.5589706520e-05	9.52124782e-06	9.7e-05 [25]

Table 2: Comparison of Exact solution for Test problem 1 with results achived

z	t	Approximate Solution	Exact solution	Value of Absolute error	Error value [25].
0.05	0.05	0.000014553615258	0.000014444806667	1.07e-09	-
0.1	0.1	0.003815423325686	0.003900097800132	7.16e-09	3.1e-04
0.2	0.2	0.017939826726911	0.018056008333945	1.34e-08	3.5e-04
0.3	0.3	0.049416386100151	0.049545686868345	1.67e-08	1.8e-04
0.4	0.4	0.034562837292023	0.034563423922912	1.62e-08	3.7e-04
0.5	0.5	0.105175840987305	0.105302640603567	1.60e-08	2.5e-04
0.6	0.6	0.192148930569310	0.192260376739845	1.24e-08	3.7e-04
0.7	0.7	0.317266350484410	0.317352402477415	7.40e-08	3.6e-04
0.8	0.8	0.487458705981956	0.487512225016511	2.80e-08	2.2e-04
0.9	0.9	0.709656476255199	0.709673351557369	2.84e-09	4.5e-04

<u>Numerical Experiment no. 2</u>:

$$\frac{\partial^2 u}{\partial t^2} - \omega \frac{\partial^2 u}{\partial z^2} + \mu\, u - \epsilon u^2 = m(z,t), \quad (z,t) \in [0,1] \times [0,T] \tag{25}$$

with initial constraints:

$u(z,0) = z,\ z \in [0,1]$

$\frac{\partial u}{\partial t}(z,0) = 0,\ z \in [0,1]$

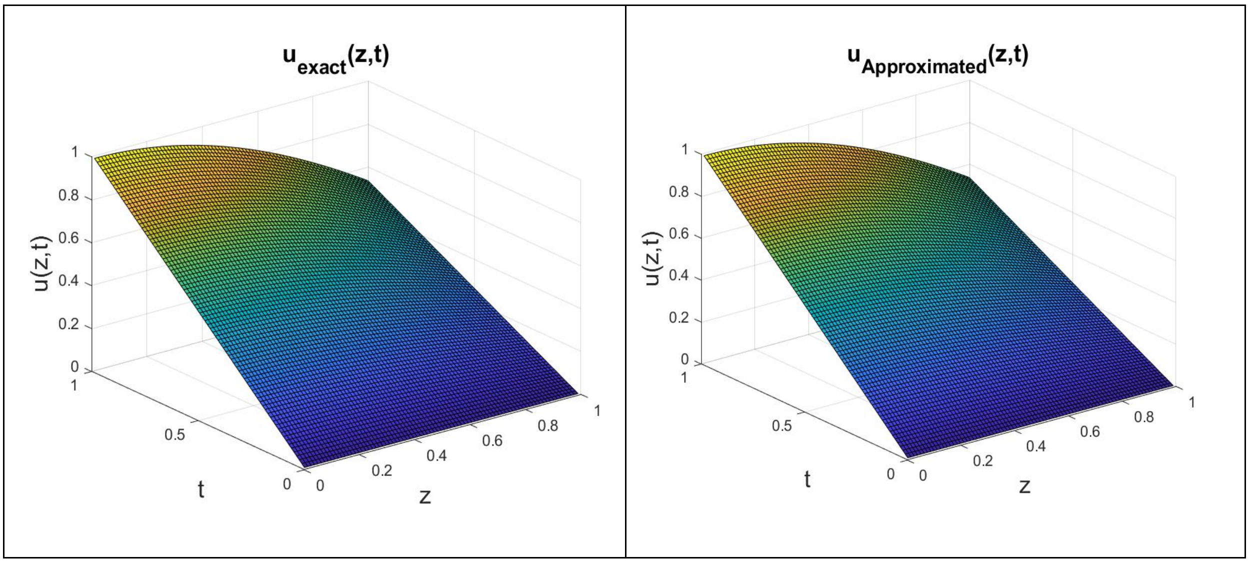
and the boundary constraints

$u(0,t) = 0, \quad t \in [0,T]$

$u(1,t) = cost, \quad t \in [0,T]$

with $\omega = -1$ and $m(z,t) = -zcost + z^2 cos^2 t$ (26)

Exact solution for the problem 2: $u(z,t) = z\, cost$ (27)



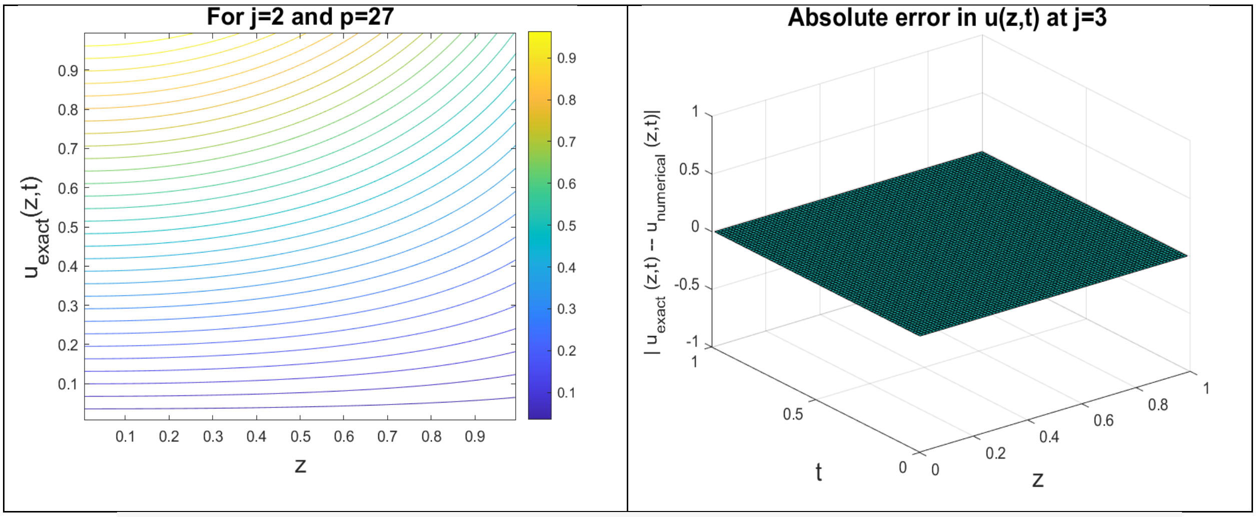


Figure 2: The solution for numerical Experiment no. 2 includes four graphical representations: exact solution, approximate solution, contour view of the exact solution, and absolute error.

Using current scheme, proposed the numerical solution in the following form:

$$u(z,t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{i\ell} \left(P_{2,i}(z) - z\, P_{2,i}(1) \right) Q_{2,\ell}(t) - z\, t\, \psi_2(1) + z\,(\xi_2(t) - \xi_2(0)) + (1-z)\big(\xi_1(t) - \xi_1(0)\big) - (1-z)\, t\psi_2(0) + t\psi_2(z) \quad (28)$$

Table 3: At different of $'j'$, value $\boldsymbol{L_2}$ and $\boldsymbol{L_\infty}$ errors for Test problem 2

Resolution level	L_∞-error (FDM)	L_∞-error (HSWM2) [16]	L_∞-error (MFDCM) [25]	L_∞-error (CM) [25]	L_∞-error (HSWM3)
3	1.1474E-01	2.2466E-08	5.7e-11	3.3e-04	0
4	1.2236E-01	5.9080E-10	5.8e-12	8.3e-05	0
5	1.2433E-01	6.4369E-11	-	-	0

Table 4: Comparison of Exact solution for Test problem 2 with results achived

z	t	Approximate Solution	Exact solution	Value of Absolute error	Value of Error [20]
0.10	0.10	0.0061388829	0.0061388829	0	$4.9593E-8$
0.20	0.20	0.0060452101	0.0060452101	0	$5.6542E-7$
0.30	0.30	0.0058926166	0.0058926166	0	$1.8493E-6$
0.40	0.40	0.0056825896	0.0056825896	0	$3.2841E-6$
0.50	0.50	0.0054171763	0.0054171763	0	$3.8373E-6$
0.60	0.60	0.0050556234	0.0050556234	0	$4.6197E-6$
0.70	0.70	0.0046817428	0.0046817428	0	$1.4141E-5$
0.80	0.80	0.0042622308	0.0042622308	0	$5.6879E-5$
0.90	0.90	0.0038011762	0.0038011762	0	$1.8571E-4$

Conclusion:

A hybrid technique called the HS-3WM method, combined with the Quasilinearisation technique, has been developed in the present study for solving non-linear KGE. The results obtained from this proposed method are found to be comparable with other existing methods, and it shows less absolute error while solving various numerical examples. The proposed technique can be applied to handle complex problems, including those whose solutions are difficult to obtain, for better results.

References:

1. M. Dehghan, A. Shokri, "Numerical solution of the nonlinear Klein Gordon equation using radial basis functions", *Journal of Computational and Applied Mathematics*, vol 230, pp. 400_410, 2009.
2. Sirendaoreji, "Auxiliary equation method and new solutions of Klein, Chaos", *Solitons and Fractals, vol.* 31, pp. 943_950, 2007.
3. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, vol. 60, Boston, Mass, USA, 1994.
4. A.M. Wazwaz, "Exact solutions to nonlinear diffusion equations obtained by the decomposition method," *Applied Mathematics and Computation*, vol. 123, no. 1, pp. 109–122, 2001.
5. A. Sadighi and D. D. Ganji, "Analytic treatment of linear and nonlinear Schrodinger equations: a study with homotopy-perturbation and Adomian decomposition methods," *Physics Letters A,* vol. 372, no. 4, pp. 465–469, 2008.
6. B. Jang, "Two-point boundary value problems by the extended Adomian decomposition method," *Journal of Computational and Applied Mathematics*, vol. 219, no. 1, pp. 253–262, 2008.
7. J. Rashidinia and M. Jokar, "Numerical solution of nonlinear Klein-Gordon equation using polynomial wavelets," *Adv. Intell. Syst. Comput.*, vol. 441, pp. 199–214, 2016.
8. A. Yildirim, "An algorithm for solving the fractional nonlinear Schrodinger equation by means of the homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 4, pp. 445–450, 2009.
9. A.-M. Wazwaz, "A study on linear and nonlinear Schrodinger equations by the variational iteration method," *Chaos, Solitons and Fractals*, vol. 37, no. 4, pp. 1136–1142, 2008.
10. Yusufoglu, E, "The variational iteration method for studying the Klein–Gordon equation," Appl. Math. Lett. 21, 669–674 2008.
11. Shiralashetti, S.C., Kumbinarasaiah, S, "Hermite wavelets operational matrix of integration for the numerical solution of nonlinear singular initial value problems," Alexandria Eng. J. 57(4), 2591–2600, 2018.
12. Shiralashetti, S.C., Kumbinarasaiah, S, "Laguerre wavelets collocation method for the numerical solution of the Benjamina Bona Mohany equations," J. Taibah Univ. Sci. 13(1), 9–15, 2019.
13. Yin, F., Tian, T., Song, J., Zhu, M, "Spectral methods using Legendre wavelets for nonlinear Klein/Sine-Gordon equations", J. Comput. Appl. Math. 275, 321–334, 2015.
14. H. Agarwal, F. Husain, and P. Saini, "Legendre Wavelet Quasilinearization Method for Nonlinear Klein-Gordon Equation with Initial Conditions," *Adv. Comput. Data Sci.*, vol. 1046, no. July, pp. 323–332, 2019.
15. V. A. Vijesh and K. H. Kumar, "Wavelet Based Numerical Simulation of Non Linear Klein/Sine Gordon Equation," *J. Comb. Inf. Syst. Sci.*, vol. 40, no. 1, pp. 225–244, 2015.
16. S. C. Shiralashetti, L. M. Angadi, A. B. Deshi and M. H. Kantli, "Haar wavelet method for the numerical solution of Klein–Gordan equations," Asian-European Journal of Mathematics, Vol. 9, No. 1,2016.
17. R. C. Mittal and S. Pandit, "New Scale-3 Haar Wavelets Algorithm for Numerical Simulation of Second Order Ordinary Differential Equations," *Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.*, vol. 89, no. 4, pp. 799–808, 2019.
18. R. C. Mittal and S. Pandit, "Sensitivity Analysis of Shock Wave Burgers ' Equation

via a Novel Algorithm Based on scale-3 Haar Wavelets," *Int. J. Comput. Math.*, vol. 95, no. 3, pp. 601–625, 2017.

19. G. Arora, R. Kumar, and H. Kaur, "A novel wavelet based hybrid method for finding the solutions of higher order boundary value problems," *Ain Shams Eng. J.*, vol. 9, no. 4, pp. 3015–3031, 2018.
20. B. Bulbul and M. Sezer, "A New Approach to Numerical Solution of Nonlinear Klein-Gordon Equation, Mathematical Problems in Engineering, 2013.
21. S. Kumbinarasaiah, "A new approach for the numerical solution for nonlinear Klein–Gordon equation", *SeMA Journal*, *77*(4), 435-456.
22. R. Kumar, "Investigation for the numerical solution of Klein-Gordon equations using scale 3 Haar wavelets", In *Journal of Physics: Conference Series*: Vol. 2267, No. 1, p. 012152, 2022.
23. R. Kumar & J. Gupta, "Numerical analysis of linear and non-linear dispersive equation using Haar scale-3 wavelet", *Mathematics in Engineering, Science & Aerospace (MESA)*, *13*(4), 2022.
24. R. Kumar, & S. Bakhtawar, "An improved algorithm based on Haar scale 3 wavelets for the numerical solution of integrodifferential equations", *Mathematics in Engineering, Science & Aerospace (MESA)*, *13*(2), 2022.
25. Lakestani, M., & Dehghan, M. (2010). Collocation and finite difference-collocation methods for the solution of nonlinear Klein–Gordon equation. *Computer Physics Communications*, *181*(8), 1392-1401.