CAYLEY GRAPH ENERGY AND CHANGE BY A SET OF GENERATORS USING EIGEN VALUES WITH A NOVEL OF APPROXIMATION OF A MOLECULE'S ENERGY

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ABSTRACT:

A finite group's abstract structure is captured in a graph known as a Cayley graph. The name-bearing Cayley's theorem, which uses a preset set of generators for the group, is used to establish the definition. In 1878, Arthur Cayley began researching Cayley graphs for finite groups. In specific applications, such as the creation of interconnection networks for parallel CPUs, Cayley graphs are used. In this study, we analyse a subset S = {b, ab, (an-1)b} for dihedral groups of order 2n, where n>= 3, and determine the Cayley graph with respect to that subset. The respected Cayley graphs' eigenvalues and energies are also calculated.Cayley graphs are graphs connected to a group and a collection of generators for that group (there is also an associated directed graph). Due to their structure and symmetry, they provide great candidates for families of expander graphs. Additionally described is the unitary Cayley graph for detecting the Euler TOTIENT energy graph.

Keywords: Eigen values, Unitary Cayley graph, Hyper-energetic graph, Energy of a graph.

I | INTRODUCTION

A Cayley graph, also termed a Cayley diagram, Cayley colour graph, group diagram, or colour group graph, encodes the abstract structure of a group. It is a vital tool in geometric and combinatorial group theory. Cayley graphs are ideal candidates to design families of expander graphs.The theorem of Cayley, which bears his name, suggests a definition for them. Let G be a straightforward, finite, undirected graph with n vertices and m edges, and let A = (aij) represent the adjacency matrix of graph G. The eigenvalues of the graph G, known as the "Spectrum of G," indicated by Spec G, are the eigenvalues λ1, λ2, . . . , λn of A, assuming in nonincreasing order. If the distinct eigenvalues of G are μ1 > μ2 > · · · > μs, and their multiplicities are m(μ1), m(μ2), . . . , m(μs), then we write

$$\text{Spec G}=\begin{pmatrix} \mu 1 & \mu 2 & \ldots\ldots & \mu s \\ m(\mu 1) & m(\mu 2) & \ldots & m(\mu s) \end{pmatrix}$$

G's specification is not contingent on G's vertices being labelled. Due to the fact that A is a real symmetric matrix with a zero trace, these eigenvalues are real and have a sum of zero. I.Gutman defined the energy E(G) of G as the sum of its eigenvalues' absolute values in 1978. Let A represent the graph's adjacency matrix, and let λ1, λ2, ... ,λn represent the graph's eigenvalues. The total absolute value of G's eigenvalues is used to define its energy.

$$\text{E(G)} = \sum_{i=1}^{n} |\lambda i|$$

A graph is said to be hyper energetic if its energy is more than the energy of the entire graph Kn, or alternatively if E(G) > 2n-2. Gutman was the first person to present this idea. Since maximally stable electron systems are related to hyper energetic molecular graphs, these graphs are significant. Theexistence of a hyper energetic graph of order n for every n = 8 has been demonstrated. If a graph is a Cayley graph on the circulant group, or if its adjacency matrix is circulant, then the graph is said to be circulant. Several results have recently been found for the hyper-energetic circulant graphs [1].
If every eigenvalue in a graph's adjacency matrix is an integer, the graph is said to be integral. Integral graphs are substantially researched and extensively examined in the literature for particular classes of integral spectrum graphs [2].

Simple eigenvalue 1 and eigenvalue 1 of multiplicity are present in the entire graph Kn n−1. Thus, E(Kn) = 2(n-1) is the formula for its energy. The order n graph G whose energy is referred to as hyper energetic and has a graph with energy, E(G) = 2(n-1) referred to as "non-hyper energetic".By using the edges of the graph G as the vertices of the line graph L(G), two vertices in L(G) are joined whenever the corresponding edges in G share a common vertex. The line graphs of all k-regular graphs, for k>= 4, are shown to be hyper energetic in [3].

The unitary Cayley graph Xn = Cay (Zn, Un) is defined for a positive integer n > 1. by the multiplicative group and the additive group of the ring Zn of integers modulo n. a unit of it. If the Zn elements are represented by the numbers 0, 1, etc. . . , n − 1, then gcd (a, n) = 1 for Un = a Zn. The vertex set for Xn is therefore V (Xn) = Zn = 0, 1. . . , n −1} and the following edge set {(a, b) : a, b ∈ Zn, gcd (a − b, n) = 1}.

The total electron energy of a particular conjugated carbon molecule is calculated as graphene energy using the Huckel theory. Due to its connection to the Gauss sum, the study of the energy of circulant graphs is of relevance in number theory. The important class of graphs known as Cayley graphs is defined by finite groups.

In this note, we compute the energy of unitary Cayley graphs and develop the necessary and sufficient conditions for Xn to be hyper energetic, which are motivated by these investigations and the findings in [13] about the energy of Knesser graphs Kn,r. We also built families of k hyper energetic non-cospectral integral circulant networks with the same number of vertices and energy for any k∈N.

Dejter first published unitary Cayley graphs in 1995, only after defining multicolouredsubgraphs of full Cayley graphs in 1990. It is particularly useful for creating interconnection networks and solving rearrangement issues. 2003 saw the advent of the Euler-TotientCayley graph and the study of some of its core aspects. Due to its colourful features and structural representation, it is one of the most significant graphs.

To identify the electron energy discovered within the Hückel atomic orbital approximation, graph energy is estimated [6, 8]. Gutman coined the term "hyper energetic graph" to describe a graph with energy greater than (2n-2). In the field of chemical graph theory, the idea of a chemical compound's energy is significant. Nikiforov

[15] expanded on the idea. By examining the correlation between the eigenvalues and singular values of an adjacency matrix for a graph G, one can determine the energy of the graph as a matrix.

The paper's structure is as follows: In Section 2, we explicitly state the formula for the energy of a unitary Cayley graph Xn and demonstrate that it is true if n has at least three unique prime factors or at least two separate prime factors. The unitary Cayley graph Xn is hyper energetic for distinct prime factors bigger than 2. Section 3 calculates the energy of Xn's complement and establishes that Xn is hyper energetic if and only if n has at least two unique prime factors and n =/ 2p, where p is a prime number. . In Section 4, we demonstrated that the integral circulant graphs Xn(pi, pj) for I =/ j are hyper energetic, with energy equal to 2k for the square-free integer n = p1,p2, ..., pk (n). In other words, we created families of k hyper energetic non-cospectral integral circulant n-vertex graphs with identical energy for every fixed k∈N.

II | ENERGY OF CAYLEY GRAPH AND EULER TOTIENT GRAPH

Let $n=p1^{a1}p2^{a2}\dots pk^{ak}$, where $p1 < p2 < \dots . < pk$ are distinct primes, and $\alpha_1 \geq 1$.

Lemma 2.1.$if\ k > 2\ or\ k = 2\ and\ p1 > 2, the\ following\ inequality\ holds$:

$$2^{k-1}\varphi(n) > n.$$

proof. For k = 1, it follows that $n = p^{\alpha}$, where p is a prime number and consequently $\phi(n) = p^{\alpha-1}(p - 1)$. For k = 2, set $n = p^{\alpha}q^{\beta}$, where p and q are distinct prime numbers. The inequality can be rewritten as

$$2 \cdot p^{\alpha-1}q^{\beta-1}(p - 1)(q - 1) > p^{\alpha}q^{\beta},$$

orpq − 2p − 2q + 2 ≤ 0. If p = 2, this inequality does not hold. For p ≥3, it follows that q ≥ 5 and therefore (p − 3)(q − 5) ≥0. After multiplication and regrouping, we get pq − 2p − 2q + 2 ≥ q + 3p − 13 ≥ 5 + 9 − 13 > 0. Thus, for k = 2 the inequality holds if and only if p > 2. Assume now that k ≥ 3. We have to prove the following inequality, equivalent to (1)

$$\left(1-\frac{1}{p1}\right)\left(1-\frac{1}{p2}\right)\dots\dots\left(1-\frac{1}{pk}\right) \geq \frac{1}{2k-1}$$

Since pi ≥ 2i − 1 for i ≥ 1, it follows 1 − 1 pi > 1

2 . In order to prove inequality, we need single estimation

$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) > \frac{1}{2}$$

for arbitrary primes 2 < p < q. This is equivalent with pq − 2p − 2q + 2 > 0, which is true according to the case k = 2. This completes the proof.

Let ω denotes a complex primitive n-th root of unity. It is proven in [27] that the eigenvalues of unitary Cayley graph Xn are

$$\lambda i = \sum_{1\le j<n, gcd(j,n)=1} \omega ij = c(i,n),\ 0\le i \le n-1.$$

The arithmetic function c(i, n) is a Ramanujan sum, and for integers i and n these sums have only integral values [28],

$$\lambda i = \varphi(n).\frac{\mu(ti)}{\varphi(ti)}\ \ and\ ti\ = \frac{n}{gcd(i,n)}.$$

where μ denotes the Möbius function.

It follows that for n ≥ 2, every nonzero eigenvalue of Xn is a divisor of ϕ(n). The nullity of a graph G (see [29]), denoted as η(G), is the multiplicity of zero as the eigenvalue.

Lemma 2.2. The nullity of Xn is n − m, where m = p1p2 · ... · pk is the maximal square-free divisor of n.

Proof. We have to count the number of solutions of the equation, $\left(\frac{n}{\mathrm{gcd}(i,n)}\right) = 0$. The number $\frac{n}{\mathrm{gcd}(i,n)}$ is square-free if and only if gcd(i,n) = $\frac{n}{l}$.

where l is an arbitrary divisor of m. The identity gcd(i, n) = d is equivalent with gcd (i/d,n/d) = 1. The number of solutions of equation gcd(i, n) = d for 1 ≤i ≤ n is equal to the number of solutions of gcd(i/d,n/d) =gcd(j,n/d) = 1, and this is exactly equal to ϕ(n/d) by definition of the Euler function. Therefore, the number of solutions of equation (2) isϕ(l). Using the well known formula$\sum k|n\ \phi(k)\ =\ n$, it follows that the nullity of Xn equals n − $\sum l|m$ϕ(l) = n – m.

Theorem 2.3. The energy of unitary Caley graph Xn equals 2^kϕ(n), where k is the number of distinct prime factors dividing n.

Proof. The energy of a graph Xn equals

$$E(Xn) = \sum_{i=1}^{n} |\lambda i| = \varphi(n)\sum_{i=1}^{n}\left|\frac{\mu\left(\frac{n}{gcd(i,n)}\right)}{\varphi\left(\frac{n}{gcd(i,n)}\right)}\right|$$

Let SF be the set of all square-free numbers. Since the absolute value of the Möbiuos function equals 0 or 1, we can reduce the sum to the square-free numbers

$$E(Xn) =\ \varphi(n)\sum_{1\le i\le n, \frac{n}{gcd(i,n)}\in SF}^{n} \frac{1}{\frac{n}{gcd(i,n)}}.$$

Therefore, we have to prove the following identity:

$2^k = \sum_{1\le i\le n, \frac{n}{gcd(i,n)}\in SF} \frac{1}{\frac{n}{gcd(i,n)}}$. The square-free numbers that divide n are exactly of the form $p_1^{\beta 1}$ p_2 β2 · ... · p_k βk , where $\beta_i = 0$ or $\beta_i = 1$. Obviously, there are 2k square-free numbers that divide n. Let m = $p_1^{\beta 1} p_2^{\beta 2}$ · ... · $p_k^{\beta k}$ be an arbitrary square-free number and let

gcd(i, n) = $p_1^{\alpha 1-\beta 1}\, p_2^{\alpha 2-\beta 2} \cdot \ldots \cdot p_k^{\alpha k-\beta k} = n/m$. (3) Analogously as in Lemma 2.2, we conclude that there are exactly ϕ(m) numbers i that satisfy the condition (3). Since the contribution of every such number equals $\frac{1}{\phi(m)}$ in the sum, we get 1 for every square-free number that divides n. This concludes the proof. Combining these two lemmas, now we can prove the main result:

Theorem 2.4. The unitary Cayley graph Xn is hyperenergetic if and only if k > 2 or k = 2 and p1 > 2.

III | COMPLEMENT OF UNITARY CAYLEY GRAPHS

The largest eigenvalue λ1 is equal to the degree of Xn, λ1 = ϕ(n). The spectra of the complement of unitary Cayley graph Xn consists of eigenvalues n − 1 − λ1, −λ2 − 1, −λ3 − 1, ... , −λn − 1 [30].

Theorem 3.1.

Let m = p1p2 · ... · pk be the largest square-free number that divides n. The energy of the complement of unitary Cayley graph Xn equals

$$E(\overline{Xn}) = 2n - (2^k-2)\varphi(n) - \prod_{i=1}^{k} pi + \prod_{i=1}^{k}(2-pi)$$

Proof. Consider the following sum:

$$S=\sum_{i=1}^{n}|\lambda i-1| = \sum_{i=1}^{n}|\lambda i+1|$$

We already know the nullity of Xn, and therefore we will sum only the non-zero eigenvalues from the spectra of Xn. Divide the sum S in two parts: when $\frac{n}{\gcd(n,i)}$ is a square-free number with an even number of divisors and when $\frac{n}{\gcd(n,i)}$ a square-free number with an odd number of divisors is. The number of even subsets of $\{p_1, p_2, \ldots, p_k\}$ is equal to the number of odd subsets of $\{p_1, p_2, \ldots, p_k\}$, since

$$\binom{k}{0} + \binom{k}{2} + \binom{k}{4} + \cdots = \binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \cdots\cdots = 2^{k-1}.$$

Using the same technique as in Theorem 2.3, in the first case we

$$\sum_{i\in Si}\left(\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})}+1\right) = \varphi(n).2^{k-1} + \sum_{I|m,\mu(1)=1}\varphi(1). \qquad (4)$$

Let l be a square-free number that divides m with an even number of prime factors. The number of solution of the equation $\frac{n}{\gcd(i,n)}$ = l is equal to ϕ(l). For all 0 ≤i< n that satisfy n = l · gcd(i, n) we have

$$\frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(i,n)}\right)} \cdot \varphi(1) + \varphi(1) = \varphi(n) + \varphi(1).$$

After taking the summation for all l|m with μ(l) = 1 we derive the identity (4). Analogously, in the second case we have

$$\sum_{i\in S2}\left(\frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(i,n)}\right)} - 1\right) = \varphi(n).2^{k-1} - \sum_{l|m,\mu(1)=-1}\varphi(1).$$

Since Euler function ϕ(n) is multiplicative, after adding the above sums we get

$$S = (n-m) + \varphi(n).2^{k} + \sum_{l|m}\mu(1)\varphi(1)$$

$$= n - m + \varphi(n).2^{k} + \prod_{i=1}^{k}\left(1-\varphi(pi)\right) = n - m + E(Xn) + \prod_{i=1}^{k}(2-pi)$$

Finally, the energy of $\overline{Xn}$ equals

$$E(\overline{Xn}) = S - |\lambda_1 - 1| + |n - \lambda_1 - 1| = n - 2\varphi(n) - 2 + n - m + E(Xn) + \prod_{i=1}^{k}(2-pi).$$

The inequality $E(\overline{Xn}) > 2n-2$ is equivalent to

$$f(n) = (2^k-2)\varphi(n) - \prod_{i=1}^{k} pi + \prod_{i=1}^{k}(2-pi) > 0$$

For the cases $n = p^{\alpha}$ or n = 2p, where p is a prime number – we have inequality f(n) < 0, and $\overline{Xn}$ is non hyperenergetic graph. If n is not a square-free number of the form $2^{\alpha}p^{\beta}$, we have

$$(2^2 - 2) \cdot 2^{\alpha-1}p^{\beta-1}(2-1)(p-1) \geq 4(p-1) > 2p,$$

providing that $\overline{Xn}$ is hyperenergetic graph. For p1 = 2 and k > 2, according to Lemma 2.1, it holds

$$(2^k-2)\,\varphi(n) > 2^{k-1}\,\varphi(n) > n \geq \prod_{i=1}^{k} pi$$

For even k > 2 or k = 2 and p1 > 2, we can use again Lemma 2.1,

$$(2^k-2)\,\varphi(n) + \prod_{i=1}^{k}(pi-2) > 2^{k-1}\,\varphi(n) > n \geq \prod_{i=1}^{k} pi$$

For odd k ≥ 3, we have similar stronger inequality as above

$$(2k-2)\prod_{i=1}^{k}(pi-1) > 2\prod_{i=1}^{k} pi > \prod_{i=1}^{k} pi + \prod_{i=1}^{k}(pi-2)$$

Using the monotonicity of $\frac{pi}{pi-1}$, it follows

$\prod_{i=1}^{k} \frac{pi}{pi-1} < \frac{3}{2} . \frac{5}{4} . (\frac{7}{6})^{k-3} < 2^{k-1} - 1.$

The last inequality is equivalent with $15 \cdot 7^{k-3} < 8 \cdot 6^{k-3} \cdot (2^{k-1} - 1)$. After regrouping, we have $8 \cdot 6^{k-3} + 15 \cdot 7^{k-3} < 32 \cdot 12^{k-3}$,

which is evidently true. Therefore, we have the following

Theorem 3.2. The complement of unitary Cayley graph Xn is hyperenergetic if and only if n has at least two distinct prime factors and n≠2p, where p is a prime number.

IV | THE PROPERTIES AND ENERGY OF THE EULER TOTIENT CAYLEY GRAPH, AS WELL AS ITS MATRIX ENERGY

Definition.: Let n be a positive integer and (Zn ⊕ n) is an additive group of integers modulo n. Let S be the set of all positive integers which are relatively prime to n and less than n. That is S ={a /1 ≤a≤ n and gcd(a, n)= 1}. Then S =φ(n), where φ is Euler totient function. The Euler totientCayley graph G (Zn , φ) is defined as the graph whose vertex set V is Zn ={0, 1, 2, 3,…….., n – 1} and the edge set E ={(x, y) /x – y ∈ S or y – x ∈ S}.

The following are basic properties of Euler totientCayley graph studied by Madhavi [14].

Lemma 2.1. The graph G (Zn, φ) is connected and simple.

Lemma 2.2. The graph G (Zn ,φ) is φ(n)- regular and its size is $\frac{n\varphi(n)}{2}$() .

Lemma 2.3. If n is prime, the graph G(Zn ,φ) is complete graph.

Lemma 2.4. If n is even, the graph G(Zn ,φ)is bipartite.

Lemma 2.5. If n ≥ 3, the graph G(Zn , φ) is Eulerian.

Lemma 2.6. The graph G(Zn, φ) is Hamiltonian.

Let G(Zn ,φ) be Euler totientCayley graph with n vertices. Let (A)=aij be the adjacency matrix of G(Zn , φ) G Zn is defined by its entries as = 1, aij if two vertices are adjacent in G (Zn , φ) and 0 otherwise and

λ1 ≥λ2 ≥λ3 ≥ ….. ≥λn are the eigen values of A(G(Zn , φ)). All eigen values with their corresponding multiplicities is the spectrum of G (Zn ,φ). The Energy of the graph is the sum of the absolute values of the eigen values of G (Zn ,φ).

That is $\varepsilon(G(Zn, \varphi))=\sum_{i=1}^{n} |\lambda i|$.

Let A (G(Zn , φ)) A(G(Zn, φ))', is a positive semi definite matrix, where A(G (Zn , φ))′ is the transpose of A(G(Zn , φ)). Let the singular values of A (G (Zn , φ)) be σ1 , σ2 , σ3 , ……., σn which are the square root values of eigen values of A (G (Zn, φ))A (G (Zn, φ))′ and these are taken in non-increasing order . $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_n$. The Matrix Energy of G(Zn ,φ) is denoted by ε_m(G(Zn ,φ)) and is defined as the summation of absolute values of singular values of G(Zn , φ).

That is $\varepsilon m\big(G(Zn,\varphi)\big) = \sum_{i=1}^{n}|\sigma i|$.

Theorem 3.3. The energy of G(Zn, φ) is 2φ(p), where p is prime.

$$A\big(G(Zn,\varphi)\big) = \begin{bmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{bmatrix} pxp.$$

Proof. Consider an Euler totientCayley graph G (Zn ,φ), with vertex set V ={0, 1, 2, ……, p − 1}, where p is prime. The adjacency matrix of G(Zn, φ) is

The characteristic equation of the above matrix is

$(\lambda+1)^{(p-1)}(\lambda-(p-1))=0$

The eigen values are –1 and (p – 1) and the corresponding multiplicities are (p – 1) and 1. Therefore, the spectrum of the graph G(Zn,φ) is $\begin{pmatrix} -1 & p-1 \\ p-1 & 1 \end{pmatrix}$

Then

$$\varepsilon(G(Z_p,\varphi)) = \sum_{i=1}^{n} |\lambda_i| = |-1|(p-1) + |(p-1)|(1) = 2(p-1).$$

Thus $\varepsilon(G(Z_p,\varphi)) = 2\varphi(p)$, where φ(p)=(p– 1), p be a prime.

Theorem 3.4: For every prime p, the matrix energy of G(Zp ,φ) is 2φ(p).

Proof. Consider an Euler totientCayley graph G(Zp , φ) with vertex set V ={0, 1, 2, ……., p − 1}, where p is prime.

From Theorem 3.1, we have

$$A\big(G(Zn,\varphi)\big) = \begin{bmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{bmatrix} pxp.$$

The transpose of the A(G (Zn ,φ))

$$A\big(G(Zn,\varphi)\big)' = \begin{bmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{bmatrix} pxp.$$

Then

$$A\big(G(Zn,\varphi)\big)A\big(G(Zn,\varphi)\big)' = \begin{bmatrix} n-1 & n-2 & n-2 & \dots & n-2 \\ n-2 & n-1 & n-2 & \dots & n-2 \\ n-2 & n-2 & n-1 & \dots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & n-2 & n-2 \dots & n-1 \end{bmatrix} pxp.$$

The characteristic equation is $(\sigma-1)^{(p-1)}(\sigma-(p-1)^2)=0$ where σ denotes the eigen value of $A(G(Z_P, \varphi))\,A(G(Z_P, \varphi))'$ and the singular value of $A(G(Zp, \varphi))$.

Then the singular values are 1 and (p – 1) and the corresponding multiplicities are (p – 1) and 1.

Therefore, the spectrum of the graph G(ZP, φ) is spec(G(ZP, φ))= $\begin{pmatrix} 1 & P-1 \\ P-1 & 1 \end{pmatrix}$. Then

$\varepsilon_m(G(Zn,\varphi))\sum_{i=1}^{n}|\sigma i| = 2(p-1)$.

Thus $\varepsilon_m(G(Zn, \varphi)) = 2\varphi(p)$ where $\varphi(p)=(p-1)$.

Theorem 3.3. The energy of $G(Z_p\alpha, \varphi)$ is $2\varphi(p^{\alpha})$, where p is prime and α> 1.

Proof. Consider the graph $G(Z_p\alpha, \varphi)$, where p is prime and α> 1.

Then the vertex set $V=\{0, 1, 2, \dots, p^{\alpha}-1\}$.

The adjacency matrix of $G(Z_p\alpha, \varphi)$ is

$$A(G(Z_{p^{\alpha}}, \varphi)) = \begin{pmatrix} Q & \dots & Q \\ \vdots & \ddots & \vdots \\ Q & \dots & Q \end{pmatrix}_{p^{\alpha}\times p^{\alpha}},$$

$$\text{where } Q = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{p\times p} \quad (p^{\alpha} times).$$

Then the characteristic equation of $A(G(Z_p\alpha, \varphi))$ is $(\lambda+p^{\alpha-1})^{(p-1)}(\lambda)^{(p\alpha-p)}(\lambda-(p^{\alpha-1})(p-1))=0$ and the eigen values are $\lambda=-p^{\alpha-1}$, 0 and $(p^{\alpha-1}-1)(p-1)$, their corresponding multiplicities are $(p-1)(p^{\alpha}-p)$ and 1.

Therefore the spectrum of $G(Z_p\alpha, \varphi)$, is

$$\begin{pmatrix} -p^{\alpha-1} & 0 & (p^{\alpha}-p)(p-1) \\ (p-1) & (p^{\alpha}-p) & 1 \end{pmatrix}.$$

Then the Energy of $G(Z_p\alpha, \varphi)$, is

$\varepsilon(G(Z_p\alpha,\varphi))=\sum_{i=1}^{n}|\lambda i|$

$$= |\ -p^{\alpha-1}\ |(p-1) + |\ 0\ |(p^{\alpha} - p)) + |\ (p^{\alpha-1})(p-1)\ |(1)$$

$$= 2p^{\alpha-1}(p-1).$$

Thus $\varepsilon(G(Z_p\alpha, \varphi)) = 2\varphi(p^{\alpha})$, where $\varphi(p^{\alpha}) = p^{\alpha-1}(p-1)$, p is prime.

Theorem 3.5:. The Matrix energy of $G(Z_p\alpha, \varphi)$, is $2\varphi(p^{\alpha})$, where p is prime and $\alpha > 1$.

Proof. Consider the graph $G(Z_p\alpha, \varphi)$, where p is prime and $\alpha > 1$. Then

the vertex set $V=\{0, 1, 2,\ldots\ldots, p^{\alpha}-1\}$.

From Theorem 3.3, the adjacency matrix of $G(Z_p\alpha, \varphi)$, is

$$A\big(G(Zn,\varphi)\big)' = \begin{bmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{bmatrix} pxp.$$

$$A(G(Z_p\alpha,\varphi)) = \begin{pmatrix} Q & \ldots & Q \\ \vdots & \ddots & \vdots \\ Q & \ldots & \ldots \end{pmatrix} p^a x p^a \text{where}$$

$$Q = \begin{bmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{bmatrix} pxp \ (p^a \text{ times}).$$

Then the transpose of $A(G(Z_p\alpha,\varphi))' = \begin{pmatrix} Q & \ldots & Q \\ \vdots & \ddots & \vdots \\ Q & \ldots & \ldots \end{pmatrix} p^a x p^a$. And

$$A(G(Z_p\alpha,\varphi))A(G(Z_p\alpha,\varphi))' = \begin{pmatrix} T & \ldots & T \\ \vdots & \ddots & \vdots \\ T & \ldots & \ldots \end{pmatrix} p^a x p^a$$

$$T = \begin{bmatrix} n-p & n-2p & n-2p & \ldots & n-2p \\ n-2p & n-p & n-2p & \ldots & n-2p \\ n-2p & n-2p & n-p & \ldots & n-2p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2p & n-2p & n-2p & n-2p \ \ldots & n-p \end{bmatrix} pxp. \ (p^a \text{ times})$$

Then the characteristic equation is

$$(\sigma + p^{a-1})^2)^{(p-1)}(\sigma)(p^a - p)(\sigma - (p^{a-1})(p-1)^2) = 0$$

So, the singular values are , $p^{\alpha-1}$,0 and ($p^{\alpha-1}$ -1)(p-1), their corresponding multiplicities are (p – 1), $(p^{\alpha}-p)$ and 1.

Therefore the spectrum of the graph $G(Z_p\alpha, \varphi)$ is

$$\begin{pmatrix} p^{\alpha-1} & 0 & (p^{\alpha}-p)(p-1) \\ (p-1) & (p^{\alpha}-p) & 1 \end{pmatrix}.$$

Then the Matrix energy of G($Z_p\alpha$, φ) is

$$\varepsilon_m(G(Z_{p^\alpha}, \varphi)) = \sum_{i=1}^{n} |\ \sigma_i\ |$$

$$= |\ p^{\alpha-1}\ |(p-1) + |\ 0\ |(p^{\alpha}-p)) + |\ (p^{\alpha-1})(p-1)\ |(1)$$

$$= 2p^{\alpha-1}(p-1).$$

Thus, $\varepsilon_m(G(Z_{p^\alpha}, \varphi)) = 2\varphi(p^{\alpha})$ where $\varphi(p^{\alpha}) = p^{\alpha-1}(p-1)$.

Thus,

$$\varepsilon_m(G(Z_{p^\alpha}, \varphi)) = 2\varphi(p^{\alpha}) \text{ where } \varphi(p^{\alpha}) = p^{\alpha-1}(p-1).$$

Theorem 3.6. The energy of G(Z_{2p} ,φ) is 4(p – 1), where p is prime.

Proof. Consider an Euler totientCayley graph G (Z_{2p} , φ) with vertex set V ={0, 1, 2, ……., 2p – 1}, where p is prime.

The adjacency matrix of G(Z_{2p} ,φ) is A(G (Z_{2p}, φ))=$\begin{pmatrix} R & S \\ S & R \end{pmatrix}$, where it's a 2p x 2p matrix.

Where R =$\begin{pmatrix} 0\ 1\ 1\ 1\ \ldots 1 \\ 1\ 0\ 1\ 1\ \ldots 1 \\ 1\ 1\ 0\ 1 \ldots 1 \\ \ldots \ldots \ldots \ldots \ldots \\ 1\ 1\ 1\ 1\ \ldots 0 \end{pmatrix} pxp \quad and\ S = \begin{pmatrix} 0\ 1\ 1\ 1\ \ldots 1 \\ 1\ 0\ 1\ 1\ \ldots 1 \\ 1\ 1\ 0\ 1 \ldots 1 \\ \ldots \ldots \ldots \ldots \ldots \\ 1\ 1\ 1\ 1\ \ldots 0 \end{pmatrix} pxp$

The characteristic equation of A (G (Z_{2p} ,φ)) is

$$(\lambda + (p-1))(\lambda+1)^{(p-1)}(\lambda-1)^{(p-1)}(\lambda-(p-1)) = 0,$$

and the eigen values are λ=–(p – 1), –1, 1 and (p – 1), their corresponding

multiplicities are 1, (p – 1), (p – 1) and 1.

Therefore, the spectrum of the graph G(Z_{2p} ,φ) is$\begin{pmatrix} -(p-1) & -1 & 1 & (p-1) \\ 1 & (p-1) & (p-1) & 1 \end{pmatrix}$

Then the Energy of G (Z_{2p} ,φ) is

$$\varepsilon(G(Z_{2p}, \varphi)) = \sum_{i=1}^{n} |\lambda_i|$$

$$= |-(p-1)|(1) + |-1|((p-1)) + |1|((p-1)) + |(p-1)|(1)$$

$$= 4(p-1).$$

Theorem 3.7. The matrix energy of $G(Z_{2p}, \varphi)$ is $4(p - 1)$, where p is prime.

Proof. From Theorem 3.5., $A(G(Z_{2p}, \varphi)) = \begin{pmatrix} R & S \\ S & R \end{pmatrix}$, where matrix is 2p x 2p.

The transpose of the above matrix is $A(G(Z_{2p}, \varphi))' = \begin{pmatrix} R & S \\ S & R \end{pmatrix}$, where matrix is 2p x 2p.

Then is $A(G(Z_{2p}, \varphi))\, A(G(Z_{2p}, \varphi))' = \begin{pmatrix} M & N \\ N & M \end{pmatrix}$ is 2p x 2p, where

$$M = \begin{pmatrix} p-2 & p-1 & 0 & \dots & p-1 \\ 0 & p-2 & 0 & \dots & 0 \\ p-1 & 0 & p-2 & \dots & p-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & p-1 & \dots & p-2 \end{pmatrix}_{pxp}$$

and

$$N = \begin{pmatrix} 0 & p-1 & 0 & \dots & 0 \\ p-1 & 0 & p-1 & \dots & p-1 \\ 0 & p-1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p-1 & 0 & \dots & 0 \end{pmatrix}_{p\times p}.$$

The characteristic equation of $A(G(Z_{2p}, \varphi))\, A(G(Z_{2p}, \varphi))'$ is $(\sigma-1)^{2(p-1)}(\sigma-(p-1)^2)^2=0$,

and the singular values of $A(G(Z_{2p}, \varphi))$ are 1, $(p - 1)$ and their corresponding multiplicities are $2(p - 1)$ and 2.

Therefore, the spectrum of the graph $G(Z_{2p}, \varphi)$ is $\begin{pmatrix} 1 & (p-1) \\ 2(p-1) & 2 \end{pmatrix}$.

Then the matrix energy of $G(Z_{2p}, \varphi)$ is

$$\varepsilon m(G(Z_{2p},\varphi)) = \sum_{i=1}^{n} |\sigma i|$$

$$= |1|(2(p-1)) + |(p-1)1|(2) = 4(p-1).$$

Theorem 3.8. For a graph $G(Z_{2p}, \varphi)$ such that $n=\prod_{i=1}^{r} pi$, where $p_1, p_2, \ldots, p_i$ are distinct primes then

$\varepsilon(G(Z_{2p}, \varphi))$ is $2^r \prod_{i=1}^{r}(pi-1)$.

Proof. Consider the graph $G(Z_{2p}, \varphi)$ with, $n = p_1, p_2, \ldots, p_i$ where $p_1, p_2, \ldots p_i$ are distinct primes and the vertex set V be $\{0, 1, 2, \ldots (p_1, p_2, \ldots p_i)-1\}$.

The adjacency matrix of G (Zn ,φ) is A(G (Zn , φ))= $\begin{pmatrix} R & S \\ S & R \end{pmatrix}_{nxn}$

Where (R) $\frac{n}{2} \times \frac{n}{2}$ and (S) $\frac{n}{2} \times \frac{n}{2}$ are the sub matrices of A(G(Zn ,φ)).

Now the characteristic equation of A(G(Zn ,φ)) is

A(G(Zn ,φ)) = $[\lambda + (p_2 - 1)(p_3 - 1),\ldots\ldots,(p_i - 1)]^{(p1 - 1)}$

$[\lambda + (p_i - 1)]^{(p_1 - 1)(p_2 - 1),\ldots\ldots,(p_{i-1} - 1)}$

$[\lambda + (p_i - 1)]^{(p_1 - 1)(p_2 - 1),\ldots\ldots,(p_{i-2} - 1)(p_i - 1)}$

⋮

$[\lambda + (p_{i-(i-2)} - 1)]^{(p_1 - 1)(p_2 - 1),\ldots\ldots,(p_{i-(i-3)} - 1)(p_{i-(i-1)} - 1),\ldots\ldots,(p_i - 1)}$

⋮

$[\lambda + (p_1 - 1)]^{(p_2 - 1)(p_2 - 1),\ldots\ldots,(p_i - 1)}$

$[\lambda - (p_1 - 1)]^{(p_2 - 1)(p_2 - 1),\ldots\ldots,(p_{i-1} - 1)}$

$[\lambda - (p_i - 1)]^{(p_1 - 1)(p_2 - 1),\ldots\ldots,(p_{i-1} - 1)}$

$[\lambda - (p_{i-1} - 1)]^{(p_1 - 1)(p_2 - 1),\ldots\ldots,(p_{i-2} - 1)(p_i - 1)}$

⋮

$[\lambda + (p_{i-(i-2)} - 1)]^{(p_1 - 1)(p_2 - 1),\ldots\ldots,(p_{i-(i-3)} - 1)(p_{i-(i-1)} - 1),\ldots\ldots,(p_i - 1)}$

⋮

$[\lambda + (p_1 - 1)]^{(p_2 - 1)(p_2 - 1),\ldots\ldots,(p_i - 1)}$

⋮

$[\lambda - (p_2 - 1)(p_3 - 1),\ldots\ldots,(p_i - 1)]^{(p1 - 1)} = 0.$

The eigen values of the above equation are,

$$\lambda = \{-(p2 - 1)(p3 - 1), \ldots, (pr - 1)],] - (pr - 1)], [-(pr - 1 - 1)], [-(pr - (r - 2) - 1], \ldots [-(p1 - 1)], [(pr - 1)], [(pr - 1 - 1)], [(pr - 1 - 1)], \ldots, [(pr - (r - 2) - 1], \ldots, [(p1 - 1)] \text{ and } [(p2 - 1)(p3 - 1), \ldots, (pr - 1)].$$

The corresponding multiplicities are

$[(p_1-1)],[(p_1-1)(p_2-1),\ldots,(p_{i-1}-1)],[(p_1-1)(p_2-1),\ldots,(p_{i-2}-1)(p_i-1)],\ldots$
$[(p_1-1)(p_2-1),\ldots,(p_{i-(i-3)}-1)(p_{i-(i-1)}-1),\ldots,(p_i-1)],\ldots,[(p_2-1)(p_2-1),\ldots,(p_i-1)],$
$[(p_1-1)(p_2-1),\ldots,(p_{i-1}-1)],[(p_1-1)(p_2-1),\ldots,(p_{i-2}-1)(p_i-1)],\ldots,[(p_1-1)$
$(p_2-1),\ldots,(p_{i-(i-3)}-1)(p_{i-(i-1)}-1),\ldots,(p_i-1)],\ldots,[(p_2-1)(p_2-1),\ldots,(p_i-1)]$
and $[(p_1-1)]$.

Hence the energy of the graph $\varepsilon(G(Z_n,\varphi))=\sum_{i=1}^{n}|\lambda i|$

$= |-(p_1-1)(p_3-1),\ldots,(p_i-1)|[(p_1-1)]$

$+|-(p_i-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-1}-1)]$

$+|-(p_{i-1}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-2}-1)(p_i-1)]$

$+$

$\vdots$

$+|-(p_{i-(i-2)}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-(i-3)}-1)(p_{i-(i-1)}-1),\ldots,(p_i-1)]$

$+$

$\vdots$

$+|-(p_1-1)|[(p_2-1),(p_2-1),\ldots,(p_i-1)]$

$+|-(p_1-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-1}-1)]$

$+|-(p_{i-1}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-2}-1)(p_i-1)]$

$+$

$\vdots$

$+|-(p_{i-(i-2)}-1)|[(p_1-1),(p_2-1),\ldots,(p_{i-(i-3)}-1)(p_{i-(i-1)}-1),\ldots,(p_i-1)]$

$+$

$\vdots$

$+|(p_1-1)|[(p_2-1),(p_2-1),\ldots,(p_i-1)]+|(p_2-1)(p_3-1),\ldots,(p_i-1)|[(p_1-1)]$

$= 2^r[(p_1-1)(p_2-1),\ldots,(p_{i-1}-1)(p_i-1)]$

$=2^r\prod_{i=1}^{r}(p_i-1).$

Observation 1.1. The graph G (Zn, φ) is Hyper energetic if and only if $n=\prod_{i=1}^{r} pi$,r≥3 where p_1, p_2,……., $p_i$are distinct primes.

Proof. In [14], the authors proved that the energy of complete graph is 2n – 2. A graph is said to be Hyper energetic, if the energy of a graph is greater than 2n – 2.

By the theorem 3.7., an Euler totientcayley graph G (Zn , φ) such that n=$\prod_{i=1}^{r} pi$, where p_1, p_2,……., p_i are distinct primes, the energy value is $2^r\prod_{i=1}^{r}(pi-1)$.

That means ε(G(Zn ,φ))> 2n – 2. Therefore the graph G(Zn , φ) such that that n=$\prod_{i=1}^{r} pi$, where p_1 p_2,………., pi are distinct primes, is Hyper energetic and vice versa.

Illustrations : Consider G(Z_{11},φ). The graph is given below.

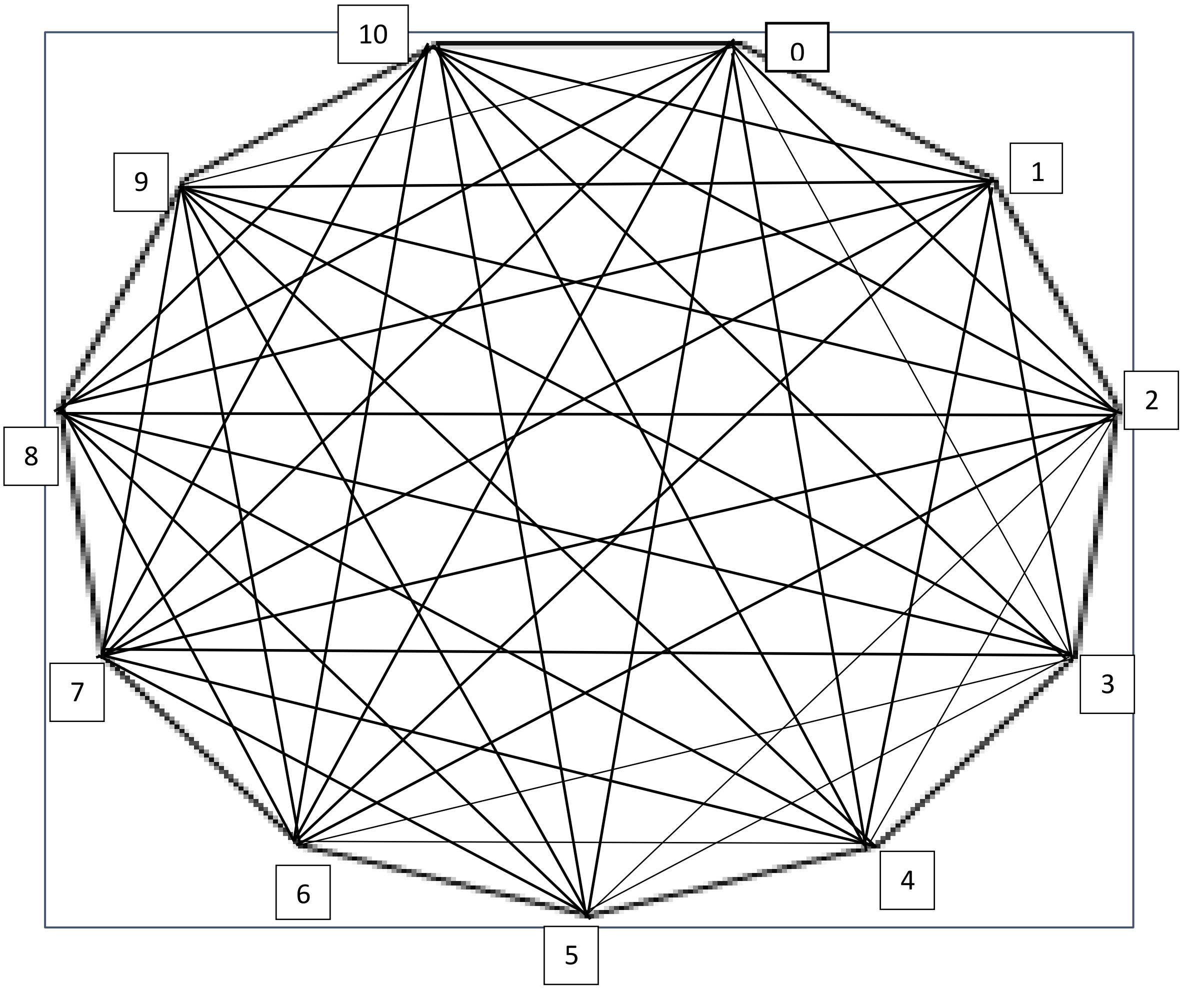


Figure:G(Z_{11},φ).

Let I={0} be the Independent Set of G(Z_{11},φ).

Then f(v)=$\begin{cases}1, & if\ v = 0\\ 0, & if\ v = 1,2,3,\ldots\ldots\ldots..,10\end{cases}$

$$=> \sum_{a\in N} f(u) = 1, \quad \forall\, v \in V\ wit\square\ f(v) > 0.$$

Thus f is an Independent Function ofG(Z_{11},φ).

Illustration 3.6: Consider G (Z_8,φ). The graph is given below.

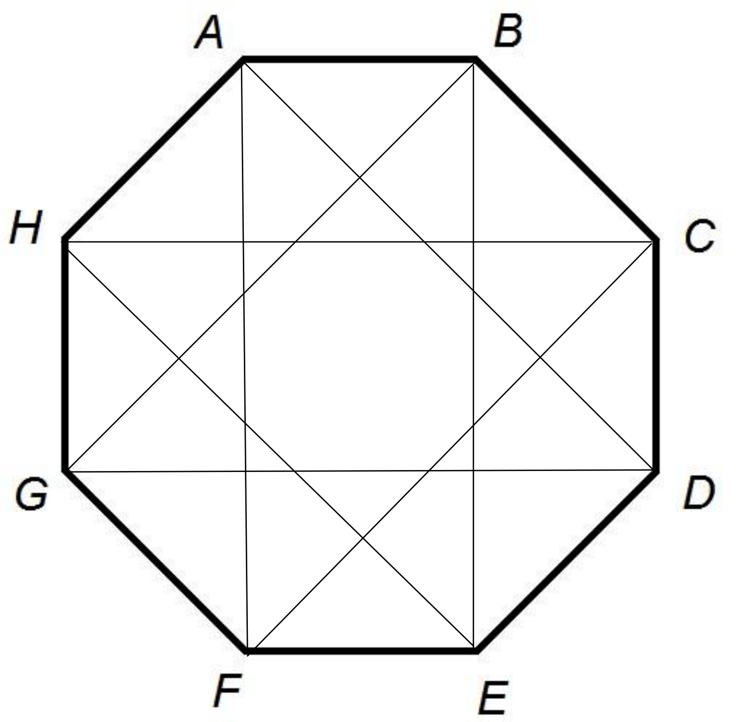


Figure : G (Z_8,φ).

The graph is | s |= 4 – regular.

Let I= {0,4} be an Independent Set of G (Z_8,φ).

Then the summation values taken over every neighbourhood N[v] of v ∈V is given below.

V:	0(A)	1(B)	2(C)	3(D)	4(E)	5(F)	6(G)	7(H)
F(v):	1	0	0	0	1	0	0	0
$\sum_{a\in N[\gamma]} f(u)$	1	2	0	2	1	2	0	2

=>$\sum_{a\in N} f(u)$=1$\forall\ v \in V$ $wit\square\ f(v) > 0$.

Hence f is an Independent Function of G (Z_8,φ).

Theorem 3.9: Let *f : V* ->[0,1] be a function defined by

f(v)=$\frac{1}{r+1}$,$\forall\ v \in V$.

Where *r* >0 denotes the degree of v∈ $V.T\square en\ f\ becomes\ an\ Independent\ Function\ of$ *G(Z_n, φ).*

Proof : Consider *G(Z_n, φ).*

Let f(v) = $\frac{1}{r+1}$,$\forall\ v \in V.,w\square ere\ r > 0\ denotes\ t\square e\ degree\ of\ t\square e\ vertex\ v$∈ V.

Case 1: Suppose n is a prime. Then every neighbourhood N[v] of v$\in V$ consists of n vertices. Then r=n-1.

Now $\sum_{n\in N[v]} f(u)=\frac{1}{r+1}+\frac{1}{r+1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.+\frac{1}{r+1}=\frac{r+1}{r+1}=1.$

=>$\sum_{u\in N[v]} f(u)=1,\ \forall\, v\in V, wit\square\ f(v)>0.$

Thus f is an Independent Function of *G(Z_n , φ).*

Case2: Suppose n is not a prime. Then G(Z_n, φ) is |s|- regular graph and |S|=r.

Now $\sum_{n\in N[v]} f(u)=\frac{1}{r+1}+\frac{1}{r+1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.+\frac{1}{r+1}=\frac{r+1}{r+1}=1.$

=>$\sum_{u\in N[v]} f(u)=1,\ \forall\, v\in V, wit\square\ f(v)>0.$

Therefore f is an Independent Function of G(Z_n, φ) for every n.

Illustration 3.8: Consider G(Z_7, φ). The graph is shown below.

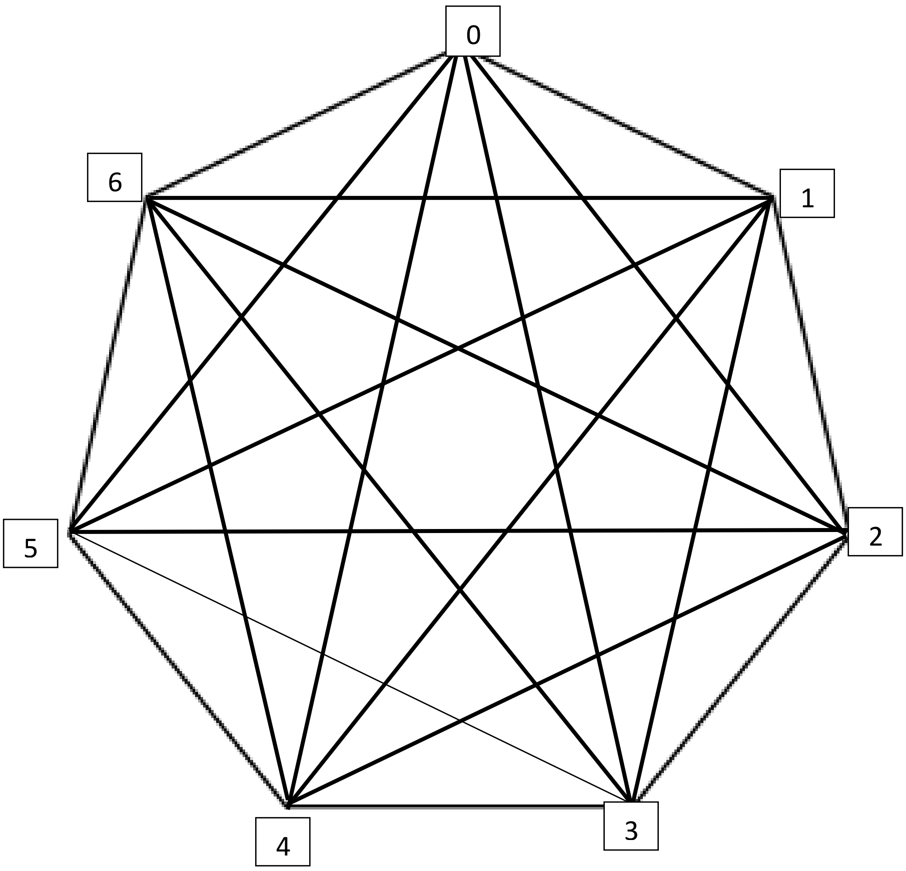


Figure: G(Z_7, φ).

Every neighbourhood N[v] of $v\ \in\ V$ consists of 6 vvertices.

Then r + 1 = 6 + 1 = 7.

Now define a function $f\ : V\rightarrow[0,1]$by

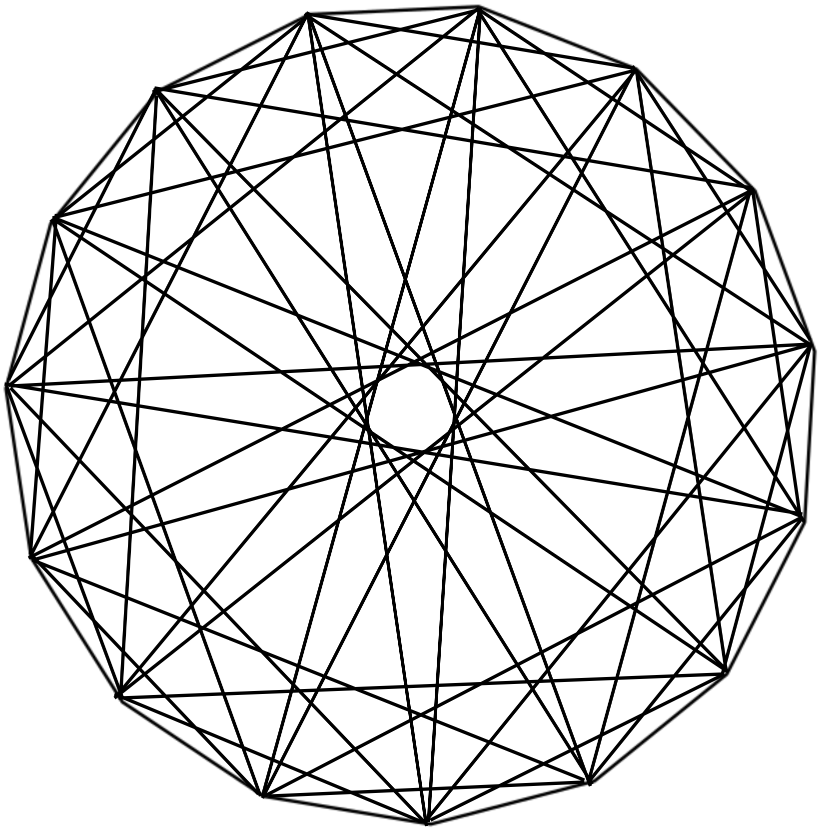
$f\,(v) = \frac{1}{7}, \forall\, v\, \in V.$

$$Then \sum_{u \in N[\,v\,]} f\,(\,u\,) = \underbrace{\frac{1}{7} + \frac{1}{7} + \cdots \;\ldots\ldots + \frac{1}{7}}_{7\ times.} = \frac{7}{7} = 1.,$$

$=> \sum_{u\in N[v]} f(u) = 1,\ \forall\, v \in V, with\ f(v) > 0.$

Thus f is an Independent Function of *G(Z₇ , φ)*.

Illustration 3.9 : Consider *G(Z₁₅ , φ)*. The graph is shown below.



It is a $|S|$ =8 – regular graph.

Then

$$Then \sum_{u \in N[\,v\,]} f\,(\,u\,) = \underbrace{\frac{1}{8} + \frac{1}{8} + \cdots \;\ldots\ldots + \frac{1}{8}}_{|S| + 1 -\ times.} = \frac{8}{8} = 1.,$$

$$\Rightarrow \sum_{u\in N\,|v\,|} f(u) = 1, \forall v \in V\ with\ f(v) > 0.$$

Thus f is an IF of *G(Z₁₅ , φ)*.

Theorem 3.10: Let f : V → [0 , 1] be a function defined by f (v) = $\begin{cases} r, & if \ v = vi \in V, \\ 1-r, & if \ v = vj \in V, \ vi \neq vj, \\ 0, & otherwise. \end{cases}$

Where 0<r<1.

Then f becomes an IF of$G(Z_n , \varphi)$, when n is a prime.

Proof: Consider $G(Z_n , \varphi)$, when n is prime. Since it is a complete graph, every neighbourhood N[v]v∈V consists of n vertices.

Then

$\sum_{u\in N[v]} f(u) = r + (1-r) + \underbrace{0+0+\cdots+0} = r + (1-r) = 1$.

$$\Rightarrow \sum_{u\in N|v|} f(u) = 1, \qquad \forall\, v \in V \ \ with \ \ f(v) > 0.$$

Thus f is an IF of $G(Z_n , \varphi)$.

Theorem3.11 : A function $f : V \to [0,1]$ is an IF of $G(Z_n , \varphi)$ if and inly if $P_f \subseteq B_f$.

Proof: Consider $G(Z_n , \varphi)$.

Suppose f: V → [0,1] is an IF of $G(Z_n , \varphi)$.

The boundary set $B_f = \{u \in V \big/ \sum_{u\in N[v]} f(u) = 1\}$.

Positive set $P_f = \{ u \in V / f(u) > 0\}$.

Let$v \in Pf$. Then $f(v) > 0$.

Since f is an IF, for all $f(v) > 0, \sum_{u\in N[v]} f(u) = 1$.

$\Rightarrow v \in Bf$.

Therefore $Pf \subseteq Bf$.

Conversely, suppose $v \in Pf$. Then $v \in Bf$, since $Pf \subseteq Bf$.

Then $\sum_{u\in N[v]} f(u) = 1, for \ f(v) > 0$.

$\Rightarrow f$ is an IF of $G(Z_n , \varphi)$.

CONCLUSION

Cayley graphs are ideal for conveying and visualising groups and their functions. There are multiple maps for disparate groups relying on the shapes to find their way. These graphs have an intrinsic spectrum and are excellently designed to simulate quantum spin networks that guarantee immaculate data change.The precise formula for the energy of a unitary Cayley graph has been examined, and it has been proven that the unitary Cayley graph is hyperenergetic. The energy and matrix energy of Euler-totientCayley graphs, as well as this graph's hyper energy, were measured.

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