Co-domination in Variant Types of Graphs

M. G. Neethialagan[1] and S. Meenakshi[2,*]

[1]Research Scholar, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies (VISTAS), Pallavaram, Chennai- 600 117, India.
Email: neethialagan15@gmail.com
[2,*]Associate Professor, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies (VISTAS), Pallavaram, Chennai- 600 117, India.
Email: meenakshikarthikeyan@yahoo.co.in

Abstract

In this paper, new results of domination in graphs called co-domination are established. Some properties of the co-dominating set are presented. Some domination parameters, such as domination γ and co-domination numbers $\bar{\gamma}$, are examined in various graph types. In this study, we analyze the co-domination number for the path, cycle, wheel, star, complete bipartite graph, web, and lollipop.

Keywords: Co-domination, bipartite, path and cycle graphs.

1. Introduction

Graph theory is merely the study of graphs, both in mathematics and computer science. Due to its many uses, graph theory study is expanding in the present. It consists of biology (genomics), operations research (scheduling), electrical engineering (communications networks, coding theory) and computer science (algorithms and computations). Graph theory concepts are utilised in picture segmentation, data mining, networking, image capture, clustering, etc. The potent combinatorial methods discovered in graph theory have also been used to establish major and well-known findings in several domains of mathematics. Similar to this, graph ideas may be used to represent network topologies. Similar to how resource allocation and scheduling use the most crucial graph colouring idea. This results in the creation of novel theorems and algorithms that have a wide range of useful applications. [1]

Graphs are simple and finite; they don't include loops. Let V be the vertex set and E be the edge set in the graph $G = (V, E)$. The study of dominant sets occupies a significant space in graph theory. As a graph theoretic notion, the dominance was initially presented by

C. Berge and O. Ore [2]. O. Ore also coined the terms dominating set and domination number. If every vertex in $V - D$ is next to a vertex in D, then the set $D \subseteq V$ is a dominant set, or $N[D] = V$. If two vertices in a simple graph G coincide on an edge in G, then the complement graph G is a graph with the same set of vertices $V(G)$ which occurs only if there is no edge between the two vertices in G [3, 4]. The domination number of G is the lowest cardinality of a dominating set D of G.

By imposing various requirements on dominating sets, many forms of dominance parameters have been investigated [3, 5]. The number of vertices that a vertex dominates is one of the criteria used to define some of the new concepts in graph dominance. Moreover, a study of a specific graph's dominance polynomial is shown in [6]. The cardinality of G is represented by the degree of v, indicated by $deg(v)$. A vertex of degree zero or one, respectively, is an independent vertex or a pendant vertex. The minimum degree (G) and maximum degree $\Delta(G)$, respectively. A regular graph is one where $\Delta(G) = \delta(G)$ [7].

The subset D of $V(G)$ is referred to as a dominant set if every vertex in $V - D$ is close to at least one vertex in D. S is referred to as a minimum dominating set if $\mathrm{S} - \{\mathrm{u}\}$ is not a dominating set for any $\mathrm{u} \in \mathrm{S}$. The dominance number $\gamma(\mathrm{G})$ of G denotes the lowest size of a dominating set of G. The complement's dominance number is the co-domination number of G. A vertex v of degree zero in G is an isolated vertex of G. In other terms, $\mathrm{V} = \{x_1 x_2 \ldots x_\mathrm{n} \colon x_\mathrm{i} \in \{0,1\}, \mathrm{i} = 1,2 \ldots \mathrm{n}\}$. If and only if x and y differ precisely in one coordinate, two vertices with the formulas $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ are connected by an edge [8]. In this paper, the concept of co-domination numbers in certain well-known graphs, including the cycle, path, wheel graph, star graph, complete bipartite graph, web graph, and lollipop graph, has been established.

2. Co-domination in Variant Graphs

Definition 2.1. Consider G to be a simple graph and D is a dominating set, the set $\overline{D}$ is called a co-dominating set if, there exists $\deg(v)$ for all $v \in V - \overline{D}$.

Definition 2.2. Let G be a simple graph and $\overline{D}$ is a co-dominating set then $\overline{D}$ is said to be a minimal co-dominating set if has no proper subset $D \subseteq \overline{D}$ is a co-dominating. We consider $\mathrm{MCDS}(G)$ refers to all minimal co-dominating sets of a graph G.

Definition 2.3. The set $\overline{D}$ is called the co-domination number if $\overline{D} = \min\{|\overline{D}_i|, \overline{D}_i \in \mathrm{MCDS}(G)\}$, and is denoted by $\bar{\gamma}(G)$.

Proposition 2.4. Let $G = (p, q)$ be a simple graph and $\overline{D}$ is a co-dominating set, then

(i) All vertices of zero, odd and even degrees belong to every co-dominating set.

(ii) $\deg(v) \geq 2$, for all $v \in V - \overline{D}$

(iii) $\gamma(G) \leq \bar{\gamma}(G)$

Proof. It is trivial from the definitions 2.1 and 2.2 of the co-dominating set.

Theorem 2.5. For any integer $n \geq 4$, the path P_n has a co-domination. Furthermore,

$$\bar{\gamma}(P_n) = \begin{cases} \frac{n}{5} + 1 & \text{if } n \equiv 0(\text{mod}5) \\ \left\lceil \frac{n}{5} \right\rceil & \text{if } n \equiv 1,2(\text{mod}5) \end{cases}$$

where $\bar{\gamma}(P_2) = 1$.

Proof. We label the P_n vertices as: $v_1, v_2, \cdots, v_n$ and let D be a set, chosen as follows:

$$D = \begin{cases} \left\{v_{5i-1}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil \right\} & \text{if } n \equiv 0,2 \ (\text{mod}5) \\ \left\{v_{5i-1}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil - 1 \right\} \cup \{v_n\} & \text{if } n \equiv 1(\text{mod}5) \end{cases}$$

Then, D is a γ-set, since it dominates all the vertices of P_n and $|D| = \left\lceil \frac{n}{5} \right\rceil$. Let $\bar{D}$ be a set such that $D \cap \bar{D} = \phi$:

$$\bar{D} = \begin{cases} \left\{v_{5i-2}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil \right\} \cup \{v_n\} & \text{if } n \equiv 0(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil - 1\right\} \cup \{1 - v_n\} & \text{if } n \equiv 1(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil \right\} & \text{if } n \equiv 2(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil \right\} \cup \{1 - v_n\} & \text{if } n \equiv 3(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil \right\} & \text{if } n \equiv 4(\text{mod}5) \end{cases}$$

Then, $\bar{D}$ is $\bar{\gamma}$-set. Hence, $\bar{\gamma}(P_n) = |\bar{D}|$ which is equal to the required identity.

Theorem 2.6. For any integer $n \geq 5$, the cycle C_n has a co-domination. Furthermore, $\bar{\gamma}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{5} \right\rceil$

Proof. Let us label the vertices of C_n as: $u_1, u_2, \cdots, u_n$. Suppose that D chosen as follows:

$$D = \begin{cases} \left\{u_{5i-1}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil \right\} & \text{if } n \equiv 0,2(\text{mod}5) \\ \left\{u_{5i-1}; i = 1,2, \dots, \left\lceil \frac{n}{5} \right\rceil - 1 \right\} \cup \{u_n\} & \text{if } n \equiv 1(\text{mod}5) \end{cases}$$

Then, D dominates all vertices of C_n and every element in it dominates two elements from $V - D$. Hence, D is γ - a set of C_n and $\gamma(C_n) = |D| = \left\lceil \frac{n}{3} \right\rceil$.

Now, we choose $\bar{D}$ according to D as follows:

$$\overline{D} = \begin{cases} \left\{v_{5i-2}; i = 1,2,\dots,\left\lceil\frac{n}{5}\right\rceil\right\} \text{ if } n \equiv 0(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2,\dots,\left\lceil\frac{n}{5}\right\rceil - 1\right\} \cup \{1 - u_n\} \text{ if } n \equiv 1(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2,\dots,\left\lceil\frac{n}{5}\right\rceil\right\} \text{ if } n \equiv 2(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2,\dots,\left\lceil\frac{n}{5}\right\rceil\right\} \cup \{1 - u_n\} \text{ if } n \equiv 3(\text{mod}5) \\ \left\{v_{5i-2}; i = 1,2,\dots,\left\lceil\frac{n}{5}\right\rceil\right\} \text{ if } n \equiv 4(\text{mod}5) \end{cases}$$

Since every $u \in \overline{D}$ dominates exactly two vertices from $V - \overline{D}$. Then, $\overline{D}$ dominates all the vertices of C_n. Hence, $\overline{D}$ is a co-dominating set of C_n and $\bar{\gamma}(C_n) = |\overline{D}| = \left\lceil\frac{n}{5}\right\rceil$.

Proposition 2.7. If G is a complete bipartite graph $K_{p,q}$, then

$$\bar{\gamma}(G) = \begin{Bmatrix} 2, & \text{if } p \text{ and } q \text{ are even} \\ p, & \text{if } p \text{ is even and } q \text{ is odd} \\ p+q, & \text{if both } p \text{ and } q \text{ are odd} \end{Bmatrix}.$$

Proof.

Suppose that V_1 and V_2 are the bipartite sets of the graph G of order p and q, respectively. Then three cases depend on p and q as follows.

Case1. It is clear that u and v dominate all other vertices in the graph as well, all other vertices with any degree. If p and q are even, then let $u \in V_1$ and $v \in V_2$. We cannot dominate this graph by a vertex. Therefore, $\bar{\gamma}(G) = 2$.

Case 2. According to Proposition 2.4(1) all the vertices in set V_1 belong to every co-dominating set, if p is even and q is odd. Therefore, $\bar{\gamma}(G) = p$, since all vertices in the set V_2 have even degree.

Case 3. By proposition 2.4(1) all vertices in the sets V_1 and V_2 belong to each co-dominating set, if p and q are odd. Thus, $\bar{\gamma}(G) = q + p$.

The result is thus obtained based on all the previous cases.

Proposition 2.8. If G is a star graph of order n, ($n \geq 3$ respectively), then $\bar{\gamma}(G) =$
$\begin{Bmatrix} n-1, \text{ if } n \equiv 1(\text{mod}2) \\ n, \ \text{ if } n \equiv 0(\text{mod}2) \end{Bmatrix}$ such that $\bar{\gamma}(\overline{S_n}) = \begin{Bmatrix} 2, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{Bmatrix}$

Proof. If G is a star graph of order n, then all pendant vertices in the star belong to every co-dominating set $\overline{D}$, as the degree of all these vertices is odd, according to proposition 2.4(2). Therefore, the remaining vertex of each graph is the centre. Following is a discussion of two cases.

Case 1. If $n \equiv 1(\mathrm{mod}2)$, then the degree of the centre vertex is even. Consequently, we include this vertex in set $V - \bar{D}$. Thus, $\bar{\gamma}(G) = n - 1$.

Case 2. If $n \equiv 0(\mathrm{mod}2)$, then the degree of the centre vertex is odd. Again, by Proposition 2.4(2), this vertex belongs to every co-dominating set. Thus, $\bar{\gamma}(G) = n$.

Therefore if $G \equiv S_n$, then $\overline{S_n} \equiv K_{n-1} \cup K_1$. Hence the result is obtained.

Proposition 2.9. If G is a wheel graph of order n, ($n \geq 4$ respectively),

then $\bar{\gamma}(G) = \begin{Bmatrix} n-1, \text{ if } n \equiv 1(\mathrm{mod}2) \\ n, \ \text{ if } n \equiv 0(\mathrm{mod}2) \end{Bmatrix}$ there exists $\bar{\gamma}(\overline{W_n}) = \begin{Bmatrix} n, \text{ if } n \text{ is odd or } n = 4 \\ 5, \text{ if } n \text{ is even } n \neq 4 \end{Bmatrix}$.

Proof. Let G be a wheel graph of order n, then by proposition 2.4(2), all vertices that lie on the cycle of the wheel belong to every co-dominating set $\bar{D}$, since the degree of all these vertices is even. Therefore, the remaining vertex of each graph is the centre. So, five cases are discussed as follows.

Case 1. If $n \equiv 1(\mathrm{mod}2)$, then the degree of the centre vertex is odd. Therefore, we place this vertex in set $V - \bar{D}$. Thus, $\bar{\gamma}(G) = n - 1$.

Case 2. If $n \equiv 0(\mathrm{mod}2)$, then the degree of the centre vertex is even. Again, by Proposition 2.4(2), this vertex belongs to every co-dominating set. Thus, $\bar{\gamma}(G) = n$.

Case 3. If $n = 4$, then $\overline{W_4} \equiv \overline{K_4}$, therefore $\bar{\gamma}(\overline{W_n}) = 4$, according to proposition 2.4(1).

Case 4. If n is odd, then the degree of all vertices in $\overline{W_n}$ are odd except for the centre vertex whose degree is zero. Therefore, according to proposition 2.4(1) $\bar{\gamma}(\overline{W_n}) = n$.

Case 5. If n is even; $n \neq 4$, then the degree of all vertices in $\overline{W_n}$ are even except for the centre vertex whose degree is zero. Clearly for any integer $n \geq 5$, the cycle C_n has a co-domination of $\bar{\gamma}(C_n) = \left\lceil \frac{n}{5} \right\rceil$, Therefore $\bar{\gamma}(\overline{W_n}) = 5$. Hence the result is obtained.

Theorem 2.10. The web graph has a co-dominating set$(\bar{D})$, $p \geq 5, q \geq 1$, $\bar{\gamma}(\mathrm{Web}_{p,q}) =$

$$\begin{cases} \frac{pq}{5} & \text{if } q \equiv 0(\mathrm{mod}5) \\ p\left\lceil \frac{q}{5} \right\rceil & \text{if } q \not\equiv 0(\mathrm{mod}5) \end{cases}$$

Proof. Let the vertices of the web graph be $V(\mathrm{Web}_{p,q}) = \{v_i^j, \mathrm{i} = 1,2, \ldots, p, \mathrm{j} = 1,2, \ldots, q\}$ and let $\bar{D} \subset V(\mathrm{Web}_{p,q})$.

If $q = 1$ we have $\mathrm{Web}_{p,q} \cong C_p$ then by Definition 2.2, we can find the minimum co-dominating set $\bar{D}$ for $\mathrm{Web}_{p,q}$. When $q \geq 2$, in this graph there are q copies of cycles $(C_p^i, i =$

$1,2,\dots,q\big)$ as shown by black lines and p copies of paths $\left(P_q^i, i = 1,2,\dots,p\ \right)$ (as shown by blue lines in Figure 1 as an example).

There are two cases depending on the order of path as follows.

Case 1. If $q \equiv 0$ (mod5), then

Let $\overline{D} = \left\{v_i^{5j-2}, i = 1,2,\dots,p\ \forall\ j = 1,2,\dots,\frac{q}{5}\right\}$. So, we dominate the graph with the vertices of $C_p^{2+5j}, j = 0,1,\dots\frac{q}{5} - 1$. Since the degree of each vertex in $\overline{D}$ is equal to 6, it dominates three vertices and is adjacent to three vertices. It is clear that $|\overline{D}| = \frac{p,q}{5}$, and $\frac{p,q}{5} \leq q\left\lceil\frac{p}{5}\right\rceil$. So, the vertices in $\overline{D}$ represent the minimum co-dominating set.

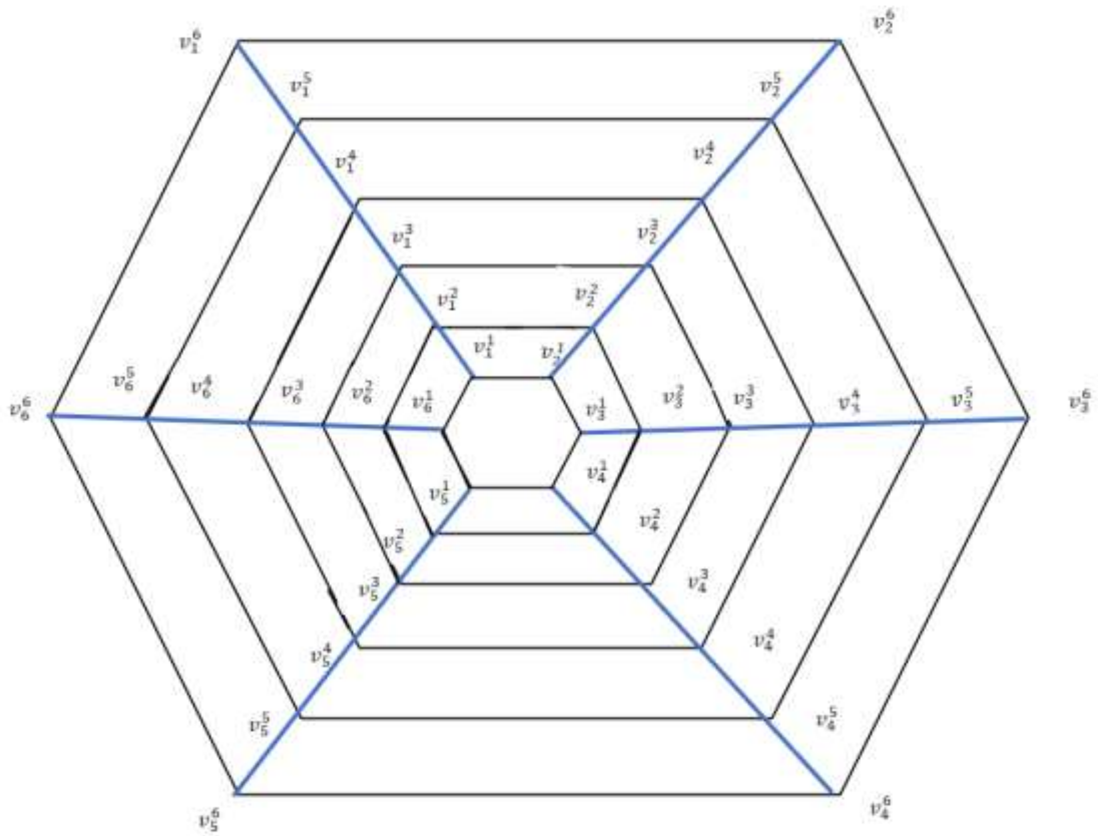

Figure 1. Web graph $\text{Web}_{6,6}$

Case 2. If $q \not\equiv 0(\text{mod } 5)$, then let

$$\overline{D} = \begin{cases} v_{5i-2}^{j}, i = 1,2,\dots,\left\lceil \frac{p}{5} \right\rceil \\ \forall j = 1,2,\dots,q \text{ if } p \equiv 0,2(\text{mod}5) \\ v_{5i-2}^{j}, i = 1,2,\dots,\left\lceil \frac{p}{5} \right\rceil - 1\} \cup \{v_{p-1}^{j}\} \\ \forall j = 1,2,\dots,q \text{ if } p \equiv 1(\text{mod } 5) \end{cases}$$

So, we dominate the graph with the vertices of $P_q^{1+3j}, j = 0,1,\dots \left\lceil \frac{q}{5} \right\rceil - 1$ if $p \equiv 0,2(\text{mod } 5)$, and $P_q^{1+5j} \cup P_q^{p-1}, j = 0,1,\dots \left\lceil \frac{p}{5} \right\rceil - 2.$ if $p \equiv 1(\text{mod } 5)$. As in Case 1 the vertices in $\overline{D}$ represent the minimum co-dominating set. Hence, we get the result.

Proposition 2.11. The (p,q)-lollipop graph has a co-dominating set $(\overline{D}), m \geq 3, n \geq 2$ then $\bar{\gamma}(L_{p,q}) = (p-2) + \left\lceil \frac{p}{3} \right\rceil$

Proof. According to the definition of a lollipop graph, the path graph P_n and the complete graph K_p common by one vertex. This is of degree 3. The fact that this vertex is a member of $V - \overline{D}$ should be noticed since, if it were a member of $\overline{D}$, it would dominate two other vertices and be next to another vertex that was a member of $\overline{D}$. The last one has degree two, thus whether it is a part of the path or the complete, it will dominate one vertex, which is contradiction. The path P_n vertices are labeled by $\{v_i, i = 1,2,\dots,n\}$ and starting from the common vertex, we began labeling the vertices in the complete graph K_p by $\{u_j, j = 1,2,\dots,p\}$ (as illustrated in Figure 2), then, Let $\overline{D} \subset V(L_{m,n})$ where $\overline{D} = \overline{D}_1 \cup \overline{D}_2$ from definition 2.2, we get the set $\overline{D}_1$, so

$$\overline{D}_1 = \begin{cases} \{v_{5i-1}, i = 1,2,\dots,\lceil p/5 \rceil\} \\ \text{if } n \equiv 0,2(\text{mod}5) \\ \{v_{5i-1}, i = 1,2,\dots,\lceil \text{p}/5 \rceil - 1\} \cup \{1 - v_n\} \\ \text{if } n \equiv 1(\text{mod}5) \end{cases}$$

To dominate the vertices in the complete graph as in Proposition 2.7 for a complete graph K_p, we have $\overline{D}_2 = \{u_j, j = 2,3,\dots,p-1\}$. We get $\overline{D}$ is the minimum co-dominating set.

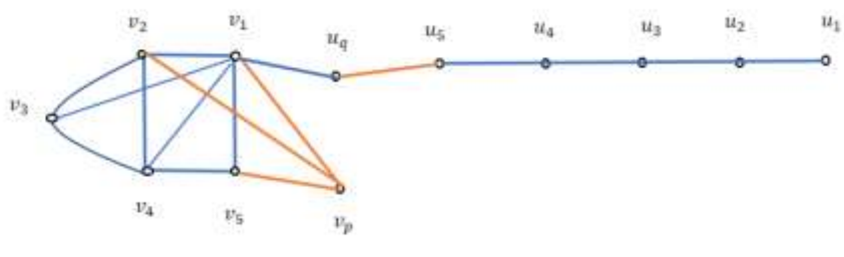

Figure 2. Lollipop graph

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