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TOPOLOGICAL INVARIANTS OF FIELD LINES

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Abstract

A topological space is generally described as a set of arbitrary elements (points) in which a concept of continuity is specified. Topological transformations preserve the neighborhood relations between mapped points and include translation, rotation, and rubber sheeting. A complete set of topological invariants allows us to recognize homeomorphism classes, which pertains to scenes of objects belonging to the same homeomorphism class. Given two scenes of geometric objects embedded in $\mathbb{R}^2$*, we can assess if they are topologically equivalent either by finding a topological transformation mapping one scene into the other or by checking whether all topological invariants are the same.*

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1. Introduction

One way to describe the subject of Topology is to say that it is qualitative geometry. The idea is that if one geometric object can be continuously transformed into another, then the two objects are to be viewed as being topologically the same. For example, a circle and a square are topologically equivalent. Physically, a rubber band can be stretched into the form of a circle or a square, as well as many other shapes which are also viewed as being topologically equivalent. On the other hand, a figure eight curve formed by two circles touching at a point is to be regarded as topologically distinct from a circle or square. A qualitative property that distinguishes the circle from the figure eight is the number of connected pieces that remain when a single point is removed: When a point is removed from a circle what remains is still connected, a single arc, whereas for a figure eight if one removes the point of contact of its two circles, what remains is two separate arcs, two separate pieces. The term used to describe two geometric objects that are topologically equivalent is homeomorphic.

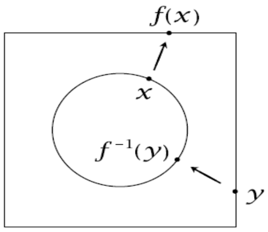

Figure 1: Homomorphism Topology

Thus, a circle and a square are homeomorphic. Concretely, if we place a circle C inside a square S with the same center point, then projecting the circle radially outward to the square defines a function $f: C \to S$, and this function is continuous: small changes in x produce small changes in f (x). The function f has an inverse $f^{-1}: S \to C$obtained by projecting the square radially inward to the circle, and this is continuous as well. One says that f is a homeomorphism between C and S. Our first goal will be to define exactly what the 'geometric objects' are that one studies in Topology. These are called topological spaces.

2. Topological Spaces

A subset O of ℝ is open if for each point x ε O there exists an interval (a, b) that contains x and is contained in O. A function $f: \mathbb{R} \to \mathbb{R}$ is continuous if for each open set O in ℝ the inverse image $f^{-1}(O): \{x \varepsilon \mathbb{R} | f(x) \varepsilon O\}$is also an open set. A topological space is a set X together with a collection O of subsets of X, called open sets, such that:

- The union of any collection of sets in O is in O.
- The intersection of any finite collection of sets in O is in O.
- Both$\phi$and X are in O.

The collection O of open sets is called a topology on X. Notice that the intersection of an infinite collection of open sets in ℝ need not be open. It is always possible to construct at least two topologies on every set X by choosing the collection O of open sets to be as large as possible or as small as possible: The collection O of all subsets of X defines a topology on X called the discrete topology. If we let O consist of just X itself and ϕ, this defines a topology, the trivial topology. Thus, we have three different topologies on ℝ, the usual topology, the discrete topology, and the trivial topology. A subset A of a topological space X is closed if its complement X - A is open.

3. Interior, Closure, and Boundary

Consider an open disk D in the plane $\mathbb{R}^2$, consisting of all the points inside a circle C . We would like to assign precise meanings to certain intuitive statements like the following:

- C is the boundary of the open disk D, and also of the closed disk $D \cup C$.
- D is the interior of the closed disk $D \cup C$, and $D \cup C$ is the closure of the open disk D.

The key distinction between points in the boundary of the disk and points in its interior is that for points in the boundary, every open set containing such a point also contains points inside the disk and points outside the disk, while each point in the interior of the disk lies in some open set entirely contained inside the disk. Given a subset A of a topological space X, then for each point x ε X exactly one of the following three possibilities holds:

(i) There exists an open set O in X with x ε O ⊂ A.
(ii) There exists an open set O in X with x ε O ⊂ X - A.
(iii) Every open set O with x ε O meets both A and X - A.

Points x such that (i) holds form a subset of A called the interior of A, written int(A). The points where (ii) hold then form int (X - A). Points x where (iii) holds forma set called the boundary or frontier of A, written ∂A. The points x where either (i) or (iii) hold are the points x such that every open set O containing x meets A. Such points are called limit points of A, and the set of these limit points is called the closure of A, written A. Note that A ⊂ A, so we have int(A) ⊂ A ⊂ A = int(A) ∪∂A, this last union being a disjoint union. We will use the symbol ∐ to denote union of disjoint subsets when we want to emphasize the disjointness, so A = int(A) ∐ ∂A and X = int(A) ∐ ∂A∐ int(X − A).

Preposition 1: For every subset A ⊂ X the following statements hold:

(a) int(A) is open.
(b) A is closed.
(c) A is open if and only if A = int(A).
(d) A is closed if and only if A = A.

Proof: (a) If x is a point in int(A) then there is an open set O_x with x ε O_x ⊂ A. We have O_x ⊂ int(A) since for each y ε O_x, O_x is an open set with y ε O_x ⊂A so y ∈ int(A). It follows that int(A) = S_x O_x , the union as x ranges over all points of int(A). This is a union of open sets and hence open.

(b) Since X = int(A) ∐ ∂A ∐ int(X - A), we have A as the complement of int(X −A), so A is closed, being the complement of an open set by part (a).

(c) If A = int(A) then A is open by (a). Conversely, suppose A is open. Then every x ε A is in int(A) since we can take O = A in condition (1). Thus A ⊂ int(A). The opposite inclusion int(A) ⊂ A always holds, so A = int(A).

(d) If A = A then A is closed by (b). Conversely, if A is closed then X - A is open, so each point of X - A is contained in an open set disjoint from A, namely the set X – A itself. This means that no point of X − A is a limit point of A, or in other words we have A ⊂ A. We always have A ⊂ A, so A = A.

Proposition 2: If B is a collection of subsets of a set X satisfying (i) and (ii) then B is a basis for a topology on X. The open sets in this topology have to be exactly the unions of sets in B since B is a basis for this topology.

Proof: Let O be the collection of subsets of X that are unions of sets in B. Obviously the union of any collection of sets in O is in O. To show the corresponding result for finite intersections it suffices by induction to show that O_1∩O_2 ε O if O_1, O_2 ε O. For each x ∈ O_1 ∩O_2 we can choose sets B_1, B_2 ε B with x ε B_1 ⊂ O_1 and x ε B_2 ⊂O_2. By (ii) there exists a set B_3 ε B with x ε B_3 ⊂ B_1 ∩ B_2 ⊂ O_1 ∩ O_2. The union of all such sets B3 as x ranges over O_1 ∩ O_2 is O_1 ∩ O_2, so O_1 ∩ O_2 ε O. Finally, X is in O by (1), and ϕ ε O since we can regard ∅ as the union of the empty collection of subsets of B.

A neighborhood of a point x in a topological space X is any set A ⊂ X that contains an open set O containing x.

4. Metric Spaces

The topology on $\mathbb{R}^n$ is defined in terms of open balls, which in turn are defined in terms of distance between points. There are many other spaces whose topology can be defined in a similar way in terms of a suitable notion of distance between points in the space. A metric space on a set X is a function d:X × X→$\mathbb{R}^n$ such that

- d(x, y) ≥0 for all $x, y \varepsilon X$ with d(x,x)=0 and d(x,y)>0 if x ≠ y.
- d(x, y)=d(y, x) for all x,y ε X
- d(x, z) ≤ d(x, y) + d(y, z) for all x,y,z ε X.

Proposition 3: The collection of all balls $B_r(x)$ for r > 0 and x ε X forms a basis for a topology on X. A topological space together with a metric that defines the topology in this way is called a metric space.

Proof: First a preliminary observation: For a point y ε $B_r(x)$ the ball $B_s(y)$ is contained in Br (x) if s ≤ r -d(x,y), since for z ε $B_s(y)$ we have d(z,y) < s and hence d(z,x) ≤ d(z,y) + d(y,x) < s + d(x,y) ≤ r

Now to show the condition to have a basis is satisfied, suppose we are given a point y ε $B_{r1}(x_1)$∩$B_{r2}(x_2)$. Then the observation in the preceding paragraph implies that $B_s(y)$ ⊂ $B_{r1}(x_1)$ ∩ B_{r2} (x_2) for any s ≤ min{r_1 − d(x_1,y), r_2 - d(x_2,y)}.

Proposition 4: The metric topology on a subset A of a metric space X is the same as the subspace topology.

Proof: Observe first that for a ball $B_r(x)$ in X, the intersection A ∩ $B_r(x)$ consists of all points in A of distance less than r from x, so this is a ball in A regarded as a metric space in itself. For a collection of such balls Br_(x_) we have $A \cap (\bigcup_\alpha B_{r\alpha}(x_\alpha)) = \bigcup_\alpha (A \cap B_{r\alpha}(x_\alpha))$. The left side of this equation is a typical open set in A with the subspace topology, and the right side is a typical open set in the metric topology, so the two topologies coincide. A subspace A ⊂ X whose subspace topology is the discrete topology is called a discrete subspace of X. This is equivalent to saying that for each point x ε A there is an open set in X whose intersection with A is just x. For example, Z is a discrete subspace of ℝ, but Q is not discrete. The sequence 1/2, 1/3, 1/4 ⋯ without its limit 0 is a discrete subspace of ℝ, but with 0 it is not discrete. For a subspace A ⊂ X, a subset of A which is open or closed in A need not be open or closed in X.

Lemma 1: For an open set A ⊂ X, a subset B ⊂ A is open in the subspace topology on A if and only if B is open in X. This is also true when "open" is replaced by "closed" throughout the statement.

Proof: If B ⊂ A is open in A, it has the form A ∩ O for some open set O in X. This intersection is open in X if A is open in X. Conversely, if B ⊂ A is open in X then A ∩ B = B is open in A. The argument for closed sets is just the same.

Lemma 2: Given a space X, a subspace Y, and a subset A ⊂ Y, then the closure of A in the space Y is the intersection of the closure of A in X with Y. This amounts to saying that a point y ε Y is a limit point of A in Y (i.e., using the subspace topology on Y) if and only if y is a limit point of A in X.

Proof: For a point y ε Y to be a limit point of A in X means that every open set O in X that contains y meets A. Since A ⊂ Y, this is equivalent to O ∩ Y meeting A, or in other words, that every open set in Y containing y meets A.

5. Continuity and Homeomorphisms

Recall the definition: A function f :X→Y between topological spaces is continuous if $f^{-1}(O)$ is open in X for each open set O in Y. For brevity, continuous functions are sometimes called maps or mappings.

Lemma 3: A function f :X→Y is continuous if and only if $f^{-1}(C)$ is closed in X for each closed set C in Y.

Proof: An evident set-theory fact is that $f^{-1}(Y - A) = X - f^{-1}(A)$ for each subset A of Y. Suppose now that f is continuous. Then for any closed set C ⊂ Y, we have Y - C open, hence the inverse image $f^{-1}(Y - C) = X - f^{-1}(C)$ is open in X, so its complement $f^{-1}(C)$ is closed. Conversely, if the inverse image of every closed set is closed, then for O open in Y the complement Y - O is closed so $f^{-1}(Y - O) = X - f^{-1}(O)$ is closed and thus $f^{-1}(O)$ is open, so f is continuous.

Lemma 4: Given a function f :X→Y and a basis B for Y, then f is continuous if and only if $f^{-1}(B)$ is open in X for each B ε B.

Proof: One direction is obvious since the sets in B are open. In the other direction, suppose $f^{-1}(B)$ is open for each B ε B. Then any open set O in Y is a union $\cup_\alpha B_\alpha$ of basis sets B_α, hence $f^{-1}(O) = f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha)$ is open in X, being a union of the open sets f^{-1}(B_).

Lemma 5: If f :X→Y and g: Y→Z are continuous, then their composition $g \circ f: X \to Z$ is also continuous.

Proof: This uses the easy set-theory fact that $(g∘f)^{-1}(A) = f^{-1}(g^{-1}(A))$ for any A ⊂Z . Thus, if f and g are continuous and A is open in Z then $g^{-1}(A)$ is open in Y so $f^{-1}(g^{-1}(A))$ is open in X. This means g∘f is continuous.

Lemma 6: If f :X→Y is continuous and A is a subspace of X, then the restriction $f|_A$ of f to A is continuous as a function A→Y.

Proof: For an open set O ⊂ Y we have $(f|_A)^{-1}(O) = f^{-1}(O) \cap A$, which is an open set in A since $f^{-1}(O)$ is open in X. A continuous map f :X→Y is a homeomorphism if it is one-to-one and onto, and its inverse function f^{-1}: Y→X is also continuous.

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6. Product Spaces

Given two sets X and Y, their product is the set X ×Y = {(x,y)|x ε X and y εY }. For example, $\mathbb{R}^2 = \mathbb{R}\, x\, \mathbb{R}$, and more generally $\mathbb{R}^m x \mathbb{R}^n = \mathbb{R}^{m+n}$. If X and Y are topological spaces, we can define a topology on X × Y by saying that a basis consists of the subsets U × V as U ranges over open sets in X and V ranges over open sets in Y. The criterion for a collection of subsets to be a basis for a topology is satisfied since $(U_1 \text{ x } V_1) \cap (U_2 \text{ x } V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. This is called the product topology on X x Y. The same topology could also be produced by taking the smaller basis consisting of products U x V where U ranges over a basis for the topology on X and V ranges over a basis for the topology on Y. This is because $(\cup_{\alpha} U_{\alpha}) \times (\cup_{\beta} V_{\beta}) = \cup_{\alpha,\beta} \, x \, (U_{\alpha} \times V_{\beta})$. For example, a basis for the product topology on $\mathbb{R} \times \mathbb{R}$consists of the open rectangles (a_1, b_1) x (a_2, b_2). This is also a basis for the usual topology on $\mathbb{R}^2$, the product topology coincides with the usual topology. More generally one can define the product X_1 x …x X_n to consist of all ordered n-tuples (x_1, … ,x_n) with $x_i \in X_i$ for each i. A basis for the product topology on X_1 x ….x X_n consists of all products U_1 x…x U_n as each U_i ranges over open sets in X_i, or just over a basis for the topology on X_i. Thus, R_n with its usual topology is also describable as the product of n copies of $\mathbb{R}$, with basis the open "boxes" (a_1, b_1) x …x (a_n, b_n).

The product $S^1 \times S^1$ is homeomorphic to a torus, say the torus T in $\mathbb{R}^3$ obtained by taking a circle C in the yz-plane disjoint from the z-axis and rotating this circle about the z-axis. We can parametrize points on T by a pair of angles (θ_1, θ_2) where θ_1 is the angle through which the yz-plane has been rotated and θ_2 is the angle between the horizontal radial vector of C pointing away from the z-axis and the radial vector to a given point of C .

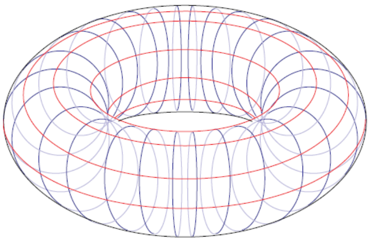

Figure 2: Homeomorphic Product Space

One can think of θ_1 and θ_2 as longitude and latitude on T. A basic open set U ×V in $S^1 \times S^1$ is a product of two open arcs, and this corresponds to an open curvilinear rectangle on T.

Proposition 5: A function f: Z→X x Y is continuous if and only if its component functions f_1: Z→X and f_2: Z→Y are both continuous.

Proof: We have $f_1 = p_1 f$ and $f_2 = p_2 f$ so f1 and f2 are continuous if f is continuous. For the converse, note that $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$, so this will be open if U and V are open and f_1 and f_2 are continuous.

7. Invariants for Relations Between Simple Lines

To exploit all invariants for lines, in this section we will concentrate on specific topological invariants characterizing the relation between pairs of simple lines. For the case of non-void intersection between two simple lines, we are going to consider all connected intersection components. To find out all invariants, it is necessary to establish an order on the points belonging to the line, hence considering the line as an oriented feature.

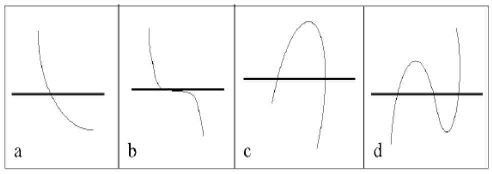

Figure 3: (a) 0-Dimensional Intersection with One Component, (b) One Dimensional Intersection with One Component, (c) 0-Dimensional Intersection with Two Components, and (d) 0-Dimensional Intersection with Three Components

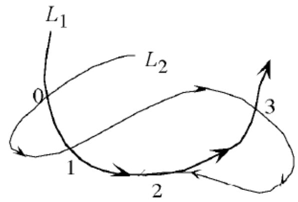

Figure 4: A Scene of Two Simple Lines with an Intersection Sequence S(L2) = (0, 1, 3, 2)

The intersection sequence describes the order in which the various components of the intersection between two lines L_1 and L_2 occur. Under a topological transformation, the intersection sequence must be preserved. Following the line L_1 from its first point and assigning numeric labels to the intersections until the last point is reached, the intersection sequence is a sequence of numbers established traversing the line L_2 and recording the labels that were previously assigned to L_1.

8. Conclusion

It is recognized that topological inconsistencies need to be removed in order to perform spatial analysis. Checking the consistency is important not only for discovering digitizing errors in data sets, but primarily for the management of multiple representations of space. The general way to assess topological equivalence would be to find a bi-continuous bijection between the two point-sets. The alternative is to show that topological invariants are preserved. The approach, which is adopted in the paper, implies as the basic step the definition of a set of topological invariants (the classifying invariant) that must satisfy two requirements: to be necessary and to uniquely identify a topological equivalence class. The classifying invariant is a description of all topological properties of any two-dimensional scene of objects and constitutes the basis for defining topological primitives.

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