Developments in Fixed Point Theory and its Applications

Zair Sajad Lone [1], Naveen Kumar [2]

[1, 2] Department of Mathematics, University Institute of Sciences, Chandigarh University, Gharuan, Mohali, Punjab, India.

Email: [1] zairlone786@gmail.com, [2] imnaveenphd@gmail.com

Abstract

This review paper discusses the developments and applications of the fixed-point theorem (FPT), which is extremely useful for both novices as well as experts in the field. This paper discusses three main theorems (i.e. Brouwer's fixed-point theorem, Tarski's fixed-point theorem, and Banach's fixed-point theorem) and their applications.

Keywords: Tarski fixed point; Banach fixed point; Fixed Point Theory (FPT); Game theory; Brouwer fixed point; metric space (m.s).

1. Introduction

In 1906 M. Frechet, proposed the concept of metric space, and it gave a basic idea of a wide range of physical, math, and other scientific structures where the concept of distance occurs. L.E.J. Brouwer provides the first fixed-point theory (FPT). H. Poincare 1886 established the fixed point. A fixed point is sometimes called invariant because it does not change its result when applying certain transformations. There are three major categories of FPT: 1. In 1912, Brouwer gives the FPT on Topology known as Topological FPT, 2. In 1922, Banach gives the FPT on Metric space known as Metric FPT, and 3. In 1955, Tarski gives FPT on Discrete space known as Discrete FPT.

Its development has been rapidly accelerating in recent years it plays a significant role in modern mathematics. The FPT is frequently approached in a variety of ways in most circumstances. As a result, it is utilized to solve issues in various disciplines of maths, such as P.D.E, variation and optimization, and probability. Many mathematicians also look for applications of their findings in sectors as diverse as economics [1], biology [2], game theory [3], chemistry [4], and so on. In the twentieth century, the scientific foundation of this theory was formed.

Banach, C Tarski, Bourbaki, Perov, Tihonov, Brouwer, Luxemburg-Jung, and Schauder theorems are all classic theorems in this theory. In addition, it is now regarded as a standard economic tool. A game theory approach is then used by economists to describe, and estimate how people behave. It has been used to study merger pricing, oligopolies, bargaining, auctions, and a variety of other topics. Economists construct Contributions to game theory through a variety of domains and interests, and they frequently combine game theory conclusions with other work. One of them, the notion of equilibrium, has shown to be extremely useful in game theory. For extending the theory of games to economics, the Noble prize in economics was awarded to J.Nash, J.Harsanyi, G.Debreu in 1983, In 2005 T.C.Schelling and R.J.Aumann, K.Arrow in 1972, and R.Selten in 1994.

2. Fixed Point Theorems

The fixed-point concept and theorem have always played a major role in a variety of fields including topology, game theory, optimal control, functional analysis, dynamics, and differential equations. Furthermore, the development of precise and efficient methods for calculating fixed points has expanded the concept's usefulness for applications, making the fixed-point method a prominent weapon in the arsenal of applied mathematicians. In recent years, methods for calculating fixed points have become more and more accurate, making them one of the most powerful tools in the arsenal of an applied mathematician.

2.1 Brouwer FPT

The result of FPT comes from Brouwer [5], who proved that in n-dimensional Euclidean space, a fixed point is contained in each self-continuous mapping of a closed unit ball.

Theorem 2.1.a: Let [0,1] be the closed unit interval and the possess of real line in the property of fixed point, i.e. for each mapping is continuous of [0,1] into itself has a fixed point.

In analysis, Theorems on topology are often applied to sequences or an infinite-dimensional space of function. Extending a theorem from a finite to infinite-dimensional space is common practice. In 1930, Schauder presented an infinite-dimensional counterpart of Brouwer's result

Theorem 2.1.b: The fixed-point property for continuous mapping exists in the nonempty compact convex subset of a normed linear space.

In the field of FPT and its applications, Brouwer and Shauder are basic theorems and the fundamental theorem of this is Brouwer's fixed point. Even though Brouwer achieved his conclusion in 1912, a slightly different version of it was proved by H. Poincare in 1886, and Bohol rediscovered it in 1904 [6]. The FPT of Brouwer has been improved and generalized in numerous ways by authors such as Schauder[7], Tychonoff[8], Kakutani[9], and many others.

2.2 Banach FPT

S. Banach (1892-1945), a pioneer of functional analysis, formulated and demonstrated the theorem in 1922 known as Banach FPT which established that integral equations have solutions [10]. Banach FPT is a useful tool in metric space which is also known as the contraction principle. It ensures the existence and uniqueness of the solution to the equation in form f(c) = c for a variety of applications f, as well as a constructive method for calculating these solutions.

Theorem 2.2.a: Assume a complete metric space. α and β are two points on them and d(α, β) represents the distance between these two points. Let T: S → S be a contraction i.e. ∃ 'c' s.t $0 \leq c < 1$ for all $\alpha, \beta \in$ S, then

$d(T\alpha, T\beta) \leq c\,d(\alpha, \beta)$

Therefore, Then T has a unique fixed point, and i.e. ∃ a unique c ∈ S s.t T(c) = c

This FPT of Banach has become a prominent tool in tackling current issues in several disciplines of mathematical analysis due to its simplicity, utility, and applications.

2.3 Tarski FPT

The FPT of Tarski can be applied to non-continuous mappings. Hence R is unbounded and closed in the interval I = [0 1], then the non-empty subset of the set I has infimum and supremum. This fact will be used to create a more general construction known as a lattice.

The lattice is a usual system Y = < F,≤ > generated by a nonempty set F, and a binary relation ≤ provide a partial order in F and that it is l.u.b and the g.l.b for any two elements $\alpha, \beta \in$ F. The relations ≥, < & > are defined as usual

Definition 2.3. I: The pair of $\alpha, \beta \in$ F inf { α, β } and sup { α, β } also ∈ F then F (partially ordered set) is known as lattice

Definition 2.3.II: If a lattice F contains the inf. and sup. of its non-empty subsets, it is said to be complete.

In 1939 Tarski [11] prove the theorem known as Tarski FPT.

Theorem 2.3.a: Let < P,≤ > be the complete lattice and E: P → P is an isotone function then the set of all fixed points of E is a nonempty complete sublattice say q. Therefore supremum {t ∈ P: E(t) ≥ t} and inf {t ∈ P: E(t) ≥ t} are also fixed points of E.

E.S. Wolk [12], A. Pasini [13], H. Hoft and M. Hoft [14], A. Pelczar [15], S. Abian and A. B. Brown [16], G. Markowsky [17], L. E. Ward Jr. [18], R. DeMarr [19], R. E. Smithson [20], J. S. W. Wong [21] have all expanded and weakened Tarski's theorem

This theorem was given constructively in 1979 by Patric Cousot and R.Cousot[22], who demonstrate that lower and upper pre-closure operators of an image of L in the set of fixed points of f. Lower and upper closure operators combine to form these pre-closure operators which are specified as stationary transfinite iteration sequence limits for F. Similarly, they have presented a constructive characterization of the set of common fixed points of a family of commuting operators, as well as the semi-continuity hypotheses. By using the Tarski-type FPT Lin Zhou [23] 1994, showed that the set of Nash equilibrium of a super-modular game is a complete lattice.

Various mappings in fixed point theory	
➢ Continuous Mappings	➢ Contraction Mappings
➢ Commuting Mappings	➢ Contractive Mappings
➢ Compatible Mappings	➢ Kannan Contraction Mappings
➢ Weakly Compatible Mapping	➢ Meir-Keeler's contraction
➢ Non-expansive Mappings	➢ Caristi's Contraction Mappings
➢ Suzuki type Mappings	➢ Ciric Contraction Mappings
➢ Expensive Mapping	➢ Geraghty Contraction Mappings
➢ Multivalued mapping	➢ Riech Contraction
➢ Lipschitz Mappings	

Various Generalizations of Metric Spaces (m.s)	
➢ Fuzzy m.s	➢ Soft m.s
➢ 2- m.s	➢ Probabilistic m.s
➢ Vector-Valued m.s	➢ Weak Partial m.s
➢ Convex m.s	➢ Modular m.s
➢ Cat (0) m.s	➢ b m.s

➢ Multiplicative m.s ➢ Complex-Valued m.s ➢ Hyperbolic space ➢ Polish space ➢ Ultra m.s ➢ Partial m.s	➢ Cone m.s ➢ G. m.s ➢ Digital m.s ➢ Generalized m.s ➢ Metric Spaces endowed with a graph

3. Some applications of Fixed-Point Theorems

This FPT is highly broad and has many applications. It plays a key role in nonlinear functional analysis problems, serving as a useful tool for establishing the uniqueness and existence of solutions of various mathematical models reflecting phenomena in diverse disciplines.

3.1 Application of Brouwer FPT

Brouwer fixed point conclusion has applications in human affairs, such as determining the best earth-to-man trajectory for space flight and analyzing, multigenerational occupational models. It can be used in Haar measures.

Piet Hein, a Danish engineer, and poet devised the game Hex in 1942[24], and John Nash rediscovered it in 1948 at Princeton. In 1952, Parker Brothers published it commercially. The game is well-liked and widely played in many countries, such as Germany, Israel, and France.

Hex, a game that many mathematicians are presumably familiar with, cannot end in a draw, which is why it leads to Brouwer's FPT. The 2-player, 2-dimensional game of Hex can be generalized naturally to an n-player, n-dimensional game, and finding an approximate fixed point of continuous mappings is a simple algorithm that can be used in both practical and theoretical settings. It can be applied to Nash equilibrium and economics.

3.2 Application of Banach FPT

Banach's principle of contraction can be used to derive the presence and uniqueness of functional relations in almost all fields of maths such as Newton Rapson iterations, signal and image reconstructions, telecommunications, ordinary differential equations, the steady-state temperature distribution, extrapolations, implicit function theorems, differential functions, artificial neural networks, tomography, inverse function theorems epidemics, a system of linear algebraic equations, integral equations, chemical reactions, Haar measures, invariant subspace problems. It is used in Google's search engine and is based on page rank algorithms

3.3 Application of Tarski FPT

FPT such as the Cantor-Bendixon theorem and Schroeder-Bernstein theorem is proved by using Tarski FPT. It contains applications and extensions to the theories of simply ordered sets and real functions, These concepts have applications in both Boolean algebra and set theory equivalency. Topology is one of the fields where it can be used. The entire derivative algebra theorems apply to any topological space. Graph theory makes use of it.

Approximation theory has made significant use of FPTs. We have data from Ky Fan [25], Brosowski [26], Hichs and Humphries [27], Sahney, Singh, and Whitfield [28], Singh and Watson [29], Reich [30], Singh [31], and Subrahmanyam [32] to prove the existence of best approximate. Brosowski [33] and Subrahmanyam [32] provide results for several types of fixed-point theorem applications (most notably Schauder's fixed-point theorem). Sahney and Singh [34] also apply the fixed-point theorem to simultaneous best approximations. We favor Brosowski [33], Chenny [35], and Dhage [36] for further references. There is typically an additional level of abstract theorems and their eventual application in approximation theory, commonly saying that under certain conditions

3.4. FPT applied to game theory. A case in point – Game in the field of quality

In certain cases, those questions can be treated as game theory problems that analyze the quality of tangible or intangible objects.

The creation of the tangible or intangible object is influenced by two sets of variables: one that raises the value of the quality index of the product, and the other lower them. As a result, the factor that enhances the quality indicator values identifies the first player, while the other set of factors decides the second player. The I^{st} player wants to produce a product of superior quality and 2^{nd} wants to produce an inferior product. The quality of the product is the outcome of their competitiveness. According to a special scientific notation**,** $A_1 = \{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m\}$ shows a group of factors that raise the value of a quality index, and by $A_2 = \{\beta_1, \ldots, \beta_k, \ldots \beta_n\}$ a series of factors that cause a reduction. Each player has a particular impact on the value of the quality indicator at a particular point of a tangible or intangible product in the life cycle. Each player selects an action α_j from A_1 and β_k from A_2. Action related to the impact of factor j on the value of the quality indicator. The possibility of the first player selecting action α_i may be expressed mathematically by a real function $f_1(\alpha_j, \beta_k)$ and its value can be read as a win for the first player. In this case, the second player's loss is represented by the function $f_2(\alpha_j,\alpha_k)$. The fact that the game's sum is null, according to specialized literature, might be written as (Owen, 1974[37])

$F_1(\alpha_j, \beta_k) + f_2(\alpha_j, \beta_k) = 0$

The main problem is how the first player selects the action α_i that reaches the maximum gain $f1(\alpha j, \beta_k)$, knowing that the other player has the same goal (von Neumann introduced the term utility, which significantly expands the term of game, implying the result of a game is not only a monetary prize but it is a vast array of events in which each player shows interest, as judged by their utility). The Nash Equilibrium is expressed in a pure strategic profile in which each player's strategy is decided by the other. As a result, the interconnections between these theories can be used to model conflict situations involving the creation of tangible or intangible high-quality products and their management.

Many writers (i.e., Lin, 2005 [38], Yu and Yang, 2004 [39]; Yang and Yu, 2002 [40]) have shown that FPT may be used to solve optimization problems, game theory problems, and Nash equilibrium problems. Cooperative and non-cooperative games are also possible. Kohlberg and Mertens in 1986, established that any finite noncooperative game has a finite set of Nash equilibrium points, one of which is required. Yu and Yang proposed in 2004, that the essential component can be used to solve nonlinear problems. There are several iteration approaches proposed by Vuong, Strodiot, and Nguyen [41] in 2012, for searching for a similar element of the set satisfying Ky Fan inequality and the set of fixed points of contraction mapping in a Hilbert space.

4. Conclusion

As generalizations and improvements of these fundamental theorems, several important conclusions on FPTs have been published. FPTs have a vast number of applications in and out of mathematics. These have been utilized regularly to verify the presence of solutions and the system's stability. Aside from these, a large number of academics are working in new spaces to construct fixed-point results as well as to extend, generalize, and enhance remaining results in a variety of domains to get interesting and useful applications of fixed-point theory

References

[1] Border, K. C. Fixed point theorems with applications to economics and game theory. Cambridge University Press, Cambridge. (1989)

[2] Turab, A. Some Applications of Fixed-Point Results in Biological Sciences: Fixed Point Theory, Banach Contraction Principle and its Applications. LAP LAMBERT Academic Publishing. (2017)

[3] Urai, K. Fixed points and economic equilibria, volume 5 of Series on Mathematical Economics and Game Theory. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. (2010)

[4] McGhee, D. F., Madbouly, N. M., and Roach, G. F. Applications of fixed-point theorems to a chemical reactor problem. In Integral methods in science and engineering (Saint Etienne, 2002) ´, pages 133–138. Birkhauser Boston, Boston, MA. (2004)

[5]. L. Brouwer, Uber abbildungen von mannigfaltigkeiten. Math. Ann. 71, 97–115. (1912)

[6] Bohl P. über die Bewegung eines mechanischen Systems in der N¨ahe einer Gleichgewichtslage, J Reine Angew. Math. 1904; 127:179-276

[7]. J. Schauder, Der fixpunktsatz in funktionalrumen. Studia Math. (1930) 2, 171–180.

[8]. A. Tychonoff, Ein Fixpunktsatz. Math. Ann. (1935) 111,767–776.

[9]. S. Kakutani, A generalization of Tychonoffs fixed point theorem. Duck Math. J. (1968) 8, 457–459.

[10] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. Fund. Math. (1922) 3, 133–181.

[11]. A. Tarski, A lattice theoretical fixpoint theorem and its applications. Pacific J. Math. (1955) 5, 285–309.

[12]. Wolk ES. Dedekind completeness and a fixed-point theorem, Canad. J Math. (1957); 9:400-405.

[13] Pasini A. Some fixed-point theorems of the mappings of partially ordered sets, Rend. Sem. Mat. Univ. Padova. (1974); 51:167- 177.

[14] Hft H, Hft M. Some fixed-point theorems for partially ordered sets, Canad. J Math. (1976); 5:992-997.

[15] Pelczar A. On the invariant points of a transformation, Ann. Polon. Math. (1961); 11:199-202.

[16] Abian S, Brown AB. A theorem on partially ordered sets with applications to fixed point theorems, Canad. J Math. (1961); 13:78-82

[17] Markowsky G. Chain-complete posets and directed sets with applications, Algebra Univ. Birkha?ser Verlag, Basel, (1976); 6:53-58
[18]. Ward JrLE. Completeness in semi-lattices, Canad. J Math. (1957); 9:578-582.
[19] DeMarr R. Common fixed points for isotone mappings, Colloquium Math. (1964); 13:45-48
[20] S mithson RE. Fixed points in partially ordered sets, Pacific J Math. (1973); 1:363-367.
[21]. Wong JSW. Common fixed points of commuting monotone mapping, Canad. J Math. (1967); 19:617-620.
[22] Patrick Cousot, Cousot R. Constructive version of Tarski's theorem, Pacific J Math. (1979); 82(1).
[23] Lin Zhou, The Set of Nash Equilibria of a Supermodular Game Is a Complete Lattice, Journal of Economic Literature. (1994); 7(2):295-300.
[24] Gale D. The game of Hex and the Brouwer fixed-point theorem The American Mathematical Monthly. 86(10):818-827.
[25] Ky Fan. Extension of two fixed point theorems of F.E. Brouder, Math. Z. (1969); 112:234-240.
[26] Brosowski B, Fixpunkt. satze in der approximations-theory, Mathematica. (1969); 11:192-220.
[27] Hicks MD. A note on fixed point theorem, J Approx. Theory. (1982); 34:221-225.
[28] Sahney BN, Singh KL, Whitefield JHM. Best approximations in locality convex spaces, J Approx. Theory. (1983); 38:182- 187.
[29] Singh SP, Watson B. Proximity maps and fixed points, J Approx. Theory. (1983); 28:72-76
[30] Riech S. Approximate selection, best approximations, fixed point, and invariant sets, J Math. Anal. Appl. (1978); 62:104-113.
[31] Singh KL. Application of fixed points to approximation theory, In Approximation Theory and Applications, Pitman Advance publishing Program, (1985), 198-211.
[32] Subrahmanyam PV. An application of a fixed-point theory to best approximation, J Approx. Theory. (1977); 20:165-175.
[33] Brosowski B. Fixed point theorems in approximation theory- theory and applications of monotone operators, Proc. NATO Advanced Study Institute, Vinice, (1968).
[34] Sahney BN, Singh SP. On best simultaneous approximations, In Approximation Theory, Academic Press. (1980), 783-789.
[35] Chenny EW. Applications of fixed point theorems to approximation theory, In Theory of Approximations with Applications, Academic Press, Ed. Law and Sahney, (1976), 1-8.
[36] Dhage BC. Applications of some fixed-point theorems in approximation theory, Math. Sci. Res. Hotline. (2000); 4:63-69.
[37] Owen, G. Teoria jocurilor, Bucureşti, Romania: Editura Tehnică (1974)..
[38] Lin, Z. Essential components of the set of weakly Pareto-nash equilibrium points for multiobjective generalized games in two different topological spaces, Journal of Optimization Theory and Applications, (2005). 124(2), 387-405.

[39] Yu, J., & Yang, H. The essential components of the set of equilibrium points for set-valued maps. J. Math. Anal. Appl. (2004)., 300, 334-342

[40] Yang, H., & Yu, J. Essential components of the set of weakly Pareto-nash equilibrium points, Applied Mathematics Letters, (2002) 15, 553-560.

[41] Vuong, P. T., Strodiot, J. J., Nguyen, V. H. Extragradient methods and line search algorithms for solving Ky Fan inequalities and fixed-point problems, Journal of Optimization Theory and Applications, (2012) 155(2), 605-627.