Boolean Filters in Pseudo-Complemented Almost Distributive Fuzzy Lattices

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Abstract. In a Pseudo-Completed Almost Distributive Fuzzy Lattice (PCADFL), the concept of Boolean filters is presented, and certain features of these filters are developed. Necessary and sufficient conditions of a proper filter to become a prime filter in PCADFL are defined. Also studied about fuzzy lattice homomorphism of Boolean filters in PCADFL. Finally, in terms of fuzzy congruences, a Boolean filter is characterized.

Keywords: Boolean filter, Pseudo-Complemented Almost Distributive Fuzzy Lattice (PCADFL), fuzzy congruence, Boolean Algebra, maximal filter.

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1. Introduction

The general development of lattice theory started by G. Birkhoff [1]. The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by George Gratzer [2] and O. Frink [13]. With this motivation, U.M. Swamy, G.C. Rao, G.N. Rao [6] introduced the concept of pseudo-complementation on an ADL. They observed that unlike in a distributive lattice, an ADL can have more than one pseudo-complementation. On the other hand, L.A. Zadeh [11] introduced Fuzzy sets to describe vagueness mathematically in its very abstractness and tried to solve such problems by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. A Fuzzy lattice as a fuzzy algebra and characterized as fuzzy sublattices in [15]. SG. Karpagavalli and A. Nasreen Sultana [9] introduced Pseudo-Complementation on Almost Distributive Fuzzy Lattices (PCADFL) and proved that it is equationally definable on ADFL by using properties of pseudo-complementation on almost distributive lattice using the fuzzy partial order relation and fuzzy lattice defined by I. Chon [12]. A. Berhanu and T. Bekalu [5] introduced ideals and filters of an ADFL analogues to the crisp concept, and the smallest ideal and the smallest filter containing a non-empty subset of R of an ADFL. In [8], M. Sambasiva Rao and K.P. Shum introduced Boolean filters in Pseudo-complemented distributive lattices and proved their properties.

The aim of this paper is to extend the concept of Boolean filters to a Pseudo-complemented ADL into PCADFL and some of the properties of these Boolean filters are derived. The class of all Boolean filters in PCADFL are characterized. A set of equivalent conditions are also derived for every PCADFL to become a Boolean algebra. Finally, the Boolean filters are characterized in terms of fuzzy congruences.

2. Preliminaries

Definition 2.1. [4] Let $(R,\ \vee,\ \wedge,\ 0)$ be an algebra of type $(2,2,0)$ and we call (R,A) is an Almost Distributive Fuzzy Lattice (ADFL) if the following condition satisfied:

(1) $A(a, a \vee 0) = A(a \vee 0, a) = 1$
(2) $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$
(3) $A\big((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)\big) = A\big((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c\big) = 1$
(4) $A\big(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)\big) = A\big((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)\big) = 1$
(5) $A\big(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)\big) = A\big((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)\big) = 1$
(6) $A\big((a \vee b) \wedge b, b\big) = A(b, (a \vee b) \wedge b) = 1$, for all $a, b, c \in R$.

Definition 2.2. [4] Let (R, A) be an ADFL. Then for any $a, b \in R$, $a \leq b$ if and only if $A(a, b) > 0$.

Definition 2.3. [2] A proper filter P of a lattice L is called a prime filter if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in L$. A proper filter M of L is called maximal if there exists no proper filter Q such that $M \subset Q$. In distributive lattice every maximal filter is a prime filter but not the converse.

Theorem 2.4. [2] Let L be a distributive lattice and $x, y \in L$ such that $x \neq y$. Then there exists a prime filter P such that $x \in P$ and $y \notin P$.

Definition 2.5. [3] An equivalence relation θ on an ADL L is called a congruence relation on L if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$, for all $(a, b), (c, d) \in \theta$.

Theorem 2.6. [3] An equivalence relation θ on an ADL L is a congruence relation if and only if for any $(a, b) \in \theta,\ x \in L, (a \vee x, b \vee x), (x \vee a, x \vee b), (a \wedge x, b \wedge x), (x \wedge a, x \wedge b)$ are all in θ.

Definition 2.7. [9] Let $(R,\ \vee,\ \wedge, 0)$ be an algebra of type $(2,2,0)$ and (R, A) be a fuzzy poset. A unary operation $a \longrightarrow a*$ on R. Then (R, A) is called a Pseudo-Complementation on Almost Distributive Fuzzy Lattice (PCADFL), if the following conditions are satisfied:

(1) $A(1, a \vee b) = A(a \vee b, 1) = 1$
(2) $A(0, a \wedge b) = A(a \wedge b, 0) = 1$
(3) $A(a \wedge a^*, 0) = A(0, a \wedge a^*) = 1$
(4) $A(a^* \wedge b,\ b) = A(b, a^* \wedge b) = 1$
(5) $A((a \vee b)^*, (a^* \wedge b^*)) = A((a^* \wedge b^*), (a \vee b)^*) = 1$
(6) $A((a^*)^*, a) = A(a, (a^*)^*) = 1$, for all $a, b \in R$.

Definition 2.8. [5] Let L be an ADFL and F be any non empty subset of R. Then F is said to be filters of an ADFL L, if it satisfies the following axioms:

(1) $a, b \in F$ implies that $a \wedge b \in F$,
(2) $a \in F,\ b \in R$ implies that $b \vee a \in F$.

Definition 2.9. [2] An element x of a pseudo-complemented lattice L is called dense if $x^* = 0$ and the set $D(L)$ of all dense element of L forms a filter of L.

Definition 2.10. [7] An ADL L with 0 is called relatively complemented if each interval $[a, b], a \leq b$, in L is a complemented lattice.

Theorem 2.11. [7] Let L be an ADL with0. Then L is relatively complemented if and only if every prime filter of L is maximal.

3. Boolean filters in PCADFLs

In this section, the definition of Boolean filters in Pseudo-Complemented Almost Distributive Fuzzy Lattice (PCADFL) is defined and some basic properties are proved.

Definition 3.1. Let (R, A) be a PCADFL, then for any filter $\mathcal{F}$ is said to be Boolean filter of a PCADFL R, if $a^* \in \mathcal{F}$, $a \in R$ implies that $a \vee a^* \in \mathcal{F}$.

$$A(a \vee a^*, a) > 0$$

Example 3.2. Let $R = \{0, a, b, c\}$ and define two binary operations $\vee$ and $\wedge$ in R as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

Cayley's table-1

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Cayley's table-2

Define a fuzzy relation in PCADFL as A: $R \times R \longrightarrow [0, 1]$ and $a^* = 0$ for all $a \neq 0$ and $0^* = x$. Clearly from cayley's table 1 & 2, (R, A) is a fuzzy poset. Then (R, A) is an

ADFL and * is a pseudo-complementation on R. Take a filter $\mathcal{F} = \{a, b, c\}$, clearly which is a Boolean filter of PCADFL of R. A filter $\mathcal{F}_1 = \{a, b\}$, which is not a Boolean filter of PCADFL R because $A(c \vee c^*, c) > 0$ as $c \vee c^* = c \notin \mathcal{F}_1$.

Lemma 3.3. Let (R, A) be a PCADFL. Then $\mathcal{D}$ is the smallest Boolean filter of R.

Proof. In general, $\mathcal{D}$ is a Boolean filter of R. Suppose that $\mathcal{B}$ is any Boolean filter of R. We prove that $A(\mathcal{D}, \mathcal{B}) > 0$. Let $a \in \mathcal{D}$ and $a^* = 0$. Since $\mathcal{B}$ is a Boolean filter of R, we get $a \vee a^* = \mathcal{B}$. Clearly, $a \vee a^* \in \mathcal{B}$. Such that $A(a \vee a^*, a) = A(a \vee 0, a) = A(a, a) > 0$. Therefore $a \in \mathcal{B}$. Hence $\mathcal{D}$ is the smallest Boolean filter of R.

Theorem 3.4. Let (R, A) be a PCADFL. Every maximal filter of R is a Boolean filter.

Proof. Suppose that (R, A) be a PCADFL and $a \in R$. Let M be a maximal filter of R. We prove that $a \vee a^* \in M$. Assume $a \vee a^* \notin M$ for any $a \in R$. Then $M \vee [a \vee a^*) = R$, which implies $x \wedge y = 0$ for any $x \in M$ and $y \in [a \vee a^*)$ such that

$$\begin{aligned} A(x \wedge y, 0) &= A(x \wedge (a \vee a^*), 0) \\ &= A\big((x \wedge a) \vee (x \wedge a^*), 0\big) \\ &= A(0 \vee (x \wedge a^*), 0) \\ &= A\big((0 \vee 0), 0\big) \\ &= A(0,0) = 1 > 0. \end{aligned}$$

Since $x \wedge a = 0$ and $x \wedge a^* = 0$. Then for any $a \in R$ and $x \in M$, $x \leq a^*$ if and only if $A(x,\ a^{**}) > 0$. Therefore $x \leq a^* \wedge a^{**}$ if and only if $A(x, a^* \wedge a^{**}) > 0$ by anti-symmetry property of A. Hence $0 \in M$, which is a contradiction to proper filter M. Such that $a \vee a^* \in M$ for all $a \in R$. Therefore M is a Boolean filter of R.

Lemma 3.5. Every prime filter is a Boolean filter of PCADFL in a relatively complemented almost distributive fuzzy lattice.

Theorem 3.6. Let (R, A) be a PCADFL. Every proper filter of a PCADFL which contains either a or a^* for all $a \in R$ is a Boolean filter of R.

Proof. Suppose $\mathcal{F}$ be a proper filter of R. Assuming the given condition either $a \in \mathcal{F}$ or $a^* \in \mathcal{F}$. We prove that $\mathcal{F}$ is maximal. Let G is a proper filter of R such that $\mathcal{F} \subsetneq G$. Suppose $x \in G - \mathcal{F}$. Since $x \in G$ and $x^* \in G$. Therefore, we get $x \wedge x^* \in G$ hence $A(x \wedge x^*, 0) > 0$ and $A(0, x \wedge x^*) > 0$ so that $x \wedge x^* = 0$. Such that $0 \in G$, which is a contradiction. Hence $\mathcal{F}$ is a maximal filter. Thus, by theorem 3.4, $\mathcal{F}$ is a Boolean filter.

Example 3.7. Let $R = \{0, p, q, r, s, 1\}$ be a PCADFL whose Hasse diagram is given in the following figure 1.

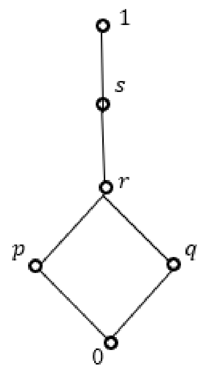

Figure 1: Hasse diagram of the PCADFL $R = \{0, p, q, r, s, 1\}$.

Consider the fuzzy lattice filters of $\mathcal{F}_1 = \{p, r, s, 1\}$; $\mathcal{F}_2 = \{q, r, s, 1\}$; $\mathcal{F}_3 = \{r, s, 1\}$; $\mathcal{F}_4 = \{s, 1\}$; and $\mathcal{F}_5 = \{1\}$. Then clearly $\mathcal{F}_1$, $\mathcal{F}_2$and $\mathcal{F}_3$ are Boolean filters of PCADFL where as $\mathcal{F}_4$ and $\mathcal{F}_5$ are not Boolean, because of $A(p \vee p^*, r) = A(r, p \vee p^*) = 1$ and $A(p \vee q, r) > 0$ by anti-symmetry property of A. Therefore $r \notin \mathcal{F}_4 \cup \mathcal{F}_5$.

Theorem 3.8. Let (R, A) be a PCADFL and $\mathcal{F}$ be a proper filter of a PCADFL if and only if the following conditions are equivalent.

(1) $\mathcal{F}$ is maximal.
(2) $a \notin \mathcal{F}$ that implies $a^* \in \mathcal{F}$ for all $a \in R$.
(3) $\mathcal{F}$ is prime Boolean filter.

Proof. (1) ⇒ (2): Let us assume $\mathcal{F}$ is a maximal filter of R. Suppose $a \in R - \mathcal{F}$ that is $a \notin \mathcal{F}$ then $A(\mathcal{F} \vee [a), R) > 0$ which implies $x \wedge a = 0$ we get $A(0, x \wedge a) = A(x \wedge a, 0) = 1$ for any $x \in \mathcal{F}$. Hence $A(a^* \wedge x, x) = A(x, a^* \wedge x) = 1$ such that $x \leq a^*$. Therefore $a^* \in \mathcal{F}$.
(2) ⇒ (3): Let $a \in R$. Suppose $a \vee a^* \notin \mathcal{F}$. Then it is clear that $a \notin \mathcal{F}$ and $a^* \notin \mathcal{F}$, which is a contradiction. Therefore $a \vee a^* \in \mathcal{F}$. Hence $\mathcal{F}$ is a Boolean filter of R. Let $a, b \in R$ with $a \vee b \in \mathcal{F}$. If $a \notin \mathcal{F}$ for all $a \in R$. Such that,

$$\begin{aligned} A(a^* \wedge b, b) &= A(0 \vee (a^* \wedge b), b) \\ &= A\big((a^* \wedge a) \vee (a^* \wedge b), b\big) \\ &= A(a^* \wedge (a \vee b), b) \\ &= A(a^* \wedge b, b) \\ &= A(b, b) = 1 > 0. \end{aligned}$$

Since $A(a^* \wedge b, b) = A(b, a^* \wedge b) = 1$ we have $a^* \wedge b \leq b$. Hence $b \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a prime Boolean filter of R.

(3) ⇒ (1): Let us assume $\mathcal{F}$ is a prime Boolean filter of R. Consider $\mathcal{F}$ is not maximal, then there exists a proper filter $\mathcal{F}'$of R such that $\mathcal{F} \subsetneq \mathcal{F}'$. Choose $a \in \mathcal{F}' - \mathcal{F}$. Since $\mathcal{F}$ is Boolean, we get $a \vee a^* \in \mathcal{F}$. Hence $\mathcal{F}$ is prime and $a \notin \mathcal{F}$, we get $a^* \in \mathcal{F} \subsetneq \mathcal{F}'$. We can conclude that $A(x \wedge x^*, 0) = A(0, x \wedge x^*) = 1 > 0$. Hence $x \wedge x^* = 0$. Such that $x \wedge x^* \in \mathcal{F}'$ we get $0 \in \mathcal{F}'$ which is a contradiction. Therefore, $\mathcal{F}$ is a maximal filter.

Theorem 3.9. Let (R,A) be a PCADFL and $\mathcal{F},G$ be two filters of PCADFL. Such that $A(\mathcal{F},G)>0$. If $\mathcal{F}$ is a Boolean filter then so is G.

Proof. Let $\mathcal{F}$ be a Boolean filter of PCADFL. Assume that G is any filter of R with $\mathcal{F}\subseteq G$. We prove that G is a Boolean filter of R. Clearly, we have $a\vee a^*\in\mathcal{F}$ for all $a\in R$. Since $\mathcal{F}\subseteq G$ we get $A(\mathcal{F},G)>0$. Therefore $a\vee a^*\in G$, for all $a\in R$. Hence G is a Boolean filter of R.

The Boolean filters are characterized in the following theorem.

Theorem 3.10. Let $\mathcal{F}$ be a proper filter of a PCADFL. Then the following conditions are equivalent.

(1) $\mathcal{F}$ is a Boolean filter.

(2) $a^{**}\in\mathcal{F}$ implies $a\in\mathcal{F}$.

(3) For $a,b\in R$, $A(b^*,a^*)=A(a^*,b^*)=1$ and $a\in\mathcal{F}$ imply $b\in\mathcal{F}$.

Proof. (1) ⇒ (2): Let us consider $\mathcal{F}$ is a Boolean filter of R. Suppose $a^{**}\in\mathcal{F}$. Since $\mathcal{F}$ is a Boolean filter of PCADFL, we get $a\vee a^*\in\mathcal{F}$. Such that,

$$\begin{aligned}A(a,(a\vee a^*)\wedge a^{**})&=A\big(a,(a^{**}\wedge a)\vee(a^*\wedge a^{**})\big)\\&=A\big(a,(a\wedge a)\vee((a^*\wedge a^{**})\big)\\&=A\big(a,a\vee((a^*\wedge a^{**})\big)\\&=A\big(a,a\vee(0\wedge a^{**})\big)\\&=A\big(a,a\vee(0\wedge a)\big)\\&=A(a,a\vee 0)\\&=A\big((a,a)\big)=1>0.\end{aligned}$$

Since $a^{**}\in\mathcal{F}$. Therefore $a\in\mathcal{F}$.

(2) ⇒ (3): Let $a,b\in R$ and $a^*=b^*$ such that $A(a^*,b^*)=A(b^*,a^*)=1>0$. Suppose $a\in\mathcal{F}$, then $a^{**}=b^{**}\in\mathcal{F}$. Hence by the condition (2) it follows that $a\in\mathcal{F}$.

(3) ⇒ (1): Assume that the condition (3). We prove that $\mathcal{F}$ is a Boolean filter of R. For that it is enough to prove that $\mathcal{D}\subseteq\mathcal{F}$ as we get $A(\mathcal{D},\mathcal{F})>0$. Let $a\in\mathcal{D}$. Then for any $x\in\mathcal{F}$, $a^*=0\leq x^*$ if and only if $A(0,x^*)>0$ by antisymmetry property of A. For any $x^{**}\in\mathcal{F}$, then $x^{**}\leq a^{**}$ if and only if $A(x^{**},a^{**})>0$ by antisymmetry property of A. Hence $a^{**}\in\mathcal{F}$. Since $a^{**}=a$ and by the condition (3), we get $a\in\mathcal{F}$. Hence $\mathcal{D}\subseteq\mathcal{F}$. Since $\mathcal{D}$ is a Boolean filter, by theorem 3.9, we get that $\mathcal{F}$ is a Boolean filter of PCADFL.

Now, we derive some results of homomorphic images in Boolean filters of PCADFLs. By a fuzzy lattice homomorphism f on a pseudo-complemented ADFL, we mean a bounded fuzzy lattice homomorphism which also preserves the pseudo-complementation, that is $A(f(a^*),f(a)^*)>0$, for all $a\in R$.

Theorem 3.11. Let $(R,\vee,\wedge,*,0,1)$ and $(R',\vee,\wedge,*,0',1')$ be any two PCADFLs and ψ a fuzzy lattice homomorphism from R onto R' if and only if it satisfies the following conditions.

(1) $\psi(\mathcal{F})$ is a Boolean filter of R' whenever $\mathcal{F}$ is a Boolean filter of R.

(2) $\psi^{-1}(G)$ is a Boolean filter of R whenever G is a Boolean filter of R'.

Proof. (1). Let $\mathcal{F}$ is a Boolean filter of R. It is known that $\psi(\mathcal{F})$ is a filter of R'. Suppose $b \in R'$. Since ψ is onto, there exists $a \in R$ such that $A(\psi(a), b) > 0$. Since $\mathcal{F}$ is a Boolean filter of R, we get $a \vee a^* \in \mathcal{F}$. Let $b = \psi(a)$, such that

$$\begin{aligned} A\big(b \vee b^*, \psi(a \vee a^*)\big) &= A\big(\psi(a) \vee (\psi(a))^*, \psi(a \vee a^*)\big) \\ &= A\big(\psi(a) \vee \psi(a^*), \psi(a \vee a^*)\big) \\ &= A\big(\psi(a \vee a^*), \psi(a \vee a^*)\big) \\ &= 1 > 0. \end{aligned}$$

Therefore $\psi(\mathcal{F})$ is a Boolean filter of R'.

(2). Suppose G be a Boolean filter of R'. It is known that $\psi^{-1}(G)$ is a filter of R. Let $a \in R$. Then $A(\psi(a \vee a^*), \psi(a) \vee \psi(a)^*) = A(\psi(a) \vee \psi(a^*), \psi(a) \vee \psi(a)^*) = A(\psi(a) \vee \psi(a)^*, \psi(a) \vee \psi(a)^*) > 0$. Clearly $\psi(a) \vee \psi(a)^* \in G$. Since $\psi(a) \in R'$. Hence, we get $a \vee a^* \in \psi^{-1}(G)$ is a Boolean filter of R.

Let (R, A) be a PCADFL and $\mathcal{F}$ a filter in R. A fuzzy congruence relation $\psi_{\mathcal{F}}$ is defined by $(a, b) \in \psi_{\mathcal{F}}$ if there exists $f \in \mathcal{F}$ such that $A(a \wedge f, b \wedge f) > 0$. Then the relation R onto R' then the set $R/\psi_{\mathcal{F}} = \{a/\psi_{\mathcal{F}} \mid a \in R\}$. It is well-known that the elements of $\mathcal{F}$ are all congruent under $\psi_{\mathcal{F}}$ and the equivalence class of $\mathcal{F}$ is the largest element in $R/\psi_{\mathcal{F}}$. It is also clear that $R/\psi_{\mathcal{F}}$ is a PCADFL.

Now, Boolean filters of PCADFL are characterized in terms of fuzzy congruence $\psi_{\mathcal{F}}$.

Theorem 3.12. Let (R, A) be a PCADFL and $\mathcal{F}$ be a filter of R if and only if the following conditions are equivalent:

(1) $\mathcal{F}$ is a Boolean filter of R.

(2) $R/\psi_{\mathcal{F}}$ is a Boolean algebra.

Proof. (1) $\Rightarrow$ (2): Let us assume that $\mathcal{F}$ is a Boolean filter of R. For any $a \in R$, we have $a \wedge a^* = 0.$ such that $A(0, a \wedge a^*) = A(0, a/\psi_{\mathcal{F}} \wedge a^*/\psi_{\mathcal{F}}) = A(0, (a \wedge a^*)/\psi_{\mathcal{F}}) = A(0, 0/\psi_{\mathcal{F}}) = A(0, 0) = 1$. Since $\mathcal{F}$ is a Boolean filter, we get that $a \vee a^* \in \mathcal{F}$. Such that $A(a/\psi_{\mathcal{F}}, a/\psi_{\mathcal{F}} \vee a^*/\psi_{\mathcal{F}}) = A(a/\psi_{\mathcal{F}}, (a \vee a^*)/\psi_{\mathcal{F}}) = A(a/\psi_{\mathcal{F}}, a/\psi_{\mathcal{F}}) = 1 > 0$ is the largest element of $R/\psi_{\mathcal{F}}$. Therefore, $R/\psi_{\mathcal{F}}$ is a Boolean algebra.

(2) $\Rightarrow$ (1): Assume that $R/\psi_{\mathcal{F}}$ is a Boolean algebra. Let $a \in R$. Then $, a/\psi_{\mathcal{F}} \in R$. Since $R/\psi_{\mathcal{F}}$ is a Boolean algebra, there exists $b \in R$ such that $A(a \wedge b, 0) = A(a/\psi_{\mathcal{F}} \wedge b/\psi_{\mathcal{F}}, 0) = A((a \wedge b)/\psi_{\mathcal{F}}, 0) = A(0/\psi_{\mathcal{F}}, 0) = A(0,0) = 1 > 0$. Consequently $(a \vee b)/\psi_{\mathcal{F}} = a/\psi_{\mathcal{F}} \vee b/\psi_{\mathcal{F}}$ is the largest element of $R/\psi_{\mathcal{F}}$. Hence $(a \wedge b, 0) \in \psi_{\mathcal{F}}$ and $a \vee b \in \mathcal{F}$. Since $(a \wedge b, 0) \in \psi_{\mathcal{F}}$, there exists $f \in \mathcal{F}$ such that $A(a \wedge b \wedge f, 0) > 0$ and thus we get $b \wedge f \leq a^*$ if and only if $A(b \wedge f, a^*) > 0$ for $f \in \mathcal{F}$. Therefore, $a \vee b \in \mathcal{F}$ and $f \in \mathcal{F}$

$$\begin{aligned} A\big((a \vee b) \wedge f, (a \vee a^*)\big) &= A\big((a \wedge f) \vee (b \wedge f), (a \vee a^*)\big) \\ &= A((a \wedge f) \vee a^*, a \vee a^*) \\ &= A\big((a \vee a^*) \wedge (f \vee a^*), a \vee a^*\big) \\ &= A\big((a \vee a^*) \wedge (f \vee 0), a \vee a^*\big) \\ &= A\big((a \vee a^*) \wedge f, a \vee a^*\big) \\ &= A(a \vee a^*, a \vee a^*) = 1 > 0. \end{aligned}$$

Hence $a \vee a^* \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a Boolean filter of R.

4. Conclusion

In this paper, we extend the concept of Boolean filters to a PCADFL. Some important necessary and sufficient conditions are established. We have shown that the dense set is the smallest Boolean filter of PCADFL. It is also observed that a Boolean filter is any prime filter of a relatively complemented PCADFL. Finally, in terms of fuzzy congruences, we eventually defined the Boolean filters. In general, it is also possible to extend fuzzy congruences in PCADFL.

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