ON THE ADJACENCY MATRIX, EIGEN VALUES AND SPECTRUM OF 3-REGULAR AND 4-REGULAR GRAPHS

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Abstract

The purpose of this paper is to show how eigenvalue is used in the field of graph theory. Earlier, the theory of eigenvalues and their maximum eigenvalue of the adjacency matrix emerging from a graph has been ascertained. Here we examine the relationship between the adjacency matrix, eigen values and spectrum of 3-regular graphs with vertices 4, 5, 6, 7 and 8. Also we have examined the relationship between the adjacency matrix, eigen values and spectrum of 4-regular graphs with vertices 5, 6, 7 and 8.

Keywords: Regular graph, eigenvalues, adjacency matrix, spectrum of graph.

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1. Introduction

Eigenvalues are one of the most essential concepts in algebra and numerous applications. The application of eigenvalues is widespread in communication system, designing bridges, designing car stereo systems, electrical engineering, mechanical engineering, machine learning and quantum computing. Among the well-known uses of eigenvalue within mathematics is determining the stability of equilibrium of a dynamical system. In this situation, the derived theory states that the equilibrium understudy is asymptotically stable if all of the eigenvalues of the Jacobian matrix of the system at equilibrium are negative [5]. Many researchers utilize eigenvalues to determine if a disease will persist or die off in the long run. This is a common epidemiological problem that epidemiologists seek to solve.

In this paper we investigated the relation between the adjacency matrix, eigenvalues and spectrum of graphs. Here we focus on studying this relation for 3-regular graphs and 4-regular graphs.

Literature Review

We give a basic definition of a graph, the adjacency matrix of a graph, rank of a matrix, eigen values of a circulant matrix and spectrum of graph.

Graph

Let $G = (V, E)$ is an ordered pair where V is the set of vertices and $v|G| = V|G|$ denoted as a number of vertices, E is the set of edges and $e|G| = E|G|$denoted as the number of edges.

Adjacency Matrix

Let $G = (V, E\ )$ be a graph with $V = [n]$. The adjacency matrix $A = A(G)$ is the $n \times n$ symmetric matrix defined by $a_{ij} = \begin{cases} 1 & \text{if } (i,j)\epsilon E, \\ 0 & \text{otherwise} \end{cases}$

Rank of Matrix The number of linearly independent rows or columns in the matrix is referred to as the matrix's rank. When all of the elements in a matrix become 0, it is said to be of rank zero. The dimension of the vector space obtained by the matrix's columns is its rank. A matrix's rank cannot be more than the number of rows or columns. The null matrix has a rank of zero.

Eigenvalues of Circulant Matrix

Let us assume that $[0, a_2, a_3, \dots, a_n]$ is the first line of the adjacency matrix. The eigenvalues of the circulant matrix in graph G is $\lambda_r = \sum_{j=2}^{n} \beta_j \omega^{(j-1)V}$ (1)

for each $r = 0,1,2, \dots, n-1$ with $\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ [2].

If A is a matrix $n \times n$, then non-zero vector x on $R^n$is said that eigenvector of A if Ax is a scalar of multiple x, Ax =λx for any scalar λ . Scalar λ is called the eigenvalue of A and x is called the eigenvector of A associated with λ [1]. To obtain the eigenvalues of the matrix n×n, rewrite Ax=λx, as $Ax = I\lambda x$ equivalently $(I\lambda - A)x = 0$ [6]. In order to be an eigenvalue must be non-zero solution from $(I\lambda - A)x = 0$ and have a non-zero solution ifet $(I\lambda - A)x = 0$, in order to obtain the characteristic equation from matrix A, the scalars that satisfy this equation are eigenvalues [6].

Spectrum of Graph

If $\lambda_0$is an eigenvalues of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the geometric multiplicity of λ_0 and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the algebraic multiplicity of λ_0[1], [3] and [4].

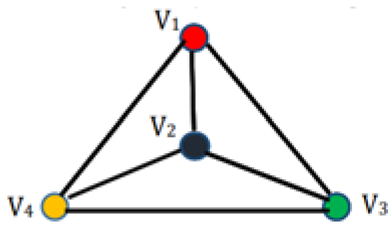
Suppose $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_s$ with $\lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{s-1}$ and $m(\lambda_0), m(\lambda_1), m(\lambda_2), \dots, m(\lambda_{s-1})$ is the multiplicities of each eigen values λ_i. The ordo matrix (2×n) that contains $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_s$in the first line and the second line is called spectrum graph G denoted by $\text{spect}(G)$. The spectrum [1], [2] of graph G can be written as:

$$\text{spect}(G) = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m(\lambda_0) & m(\lambda_1) & m(\lambda_2) & \dots & m(\lambda_{s-1}) \end{bmatrix}. \quad (2)$$

2. Result and Discussion

Regular Graph with 4 Vertices

The adjacency matrix of the 3-Regular graph with 4 vertices is as follows:



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Determine eigenvalues of a circulant matrix:

Suppose n=4 and V=1, 2, 3, 4, we can find the eigenvalues using the equation (1).
For V=1

$\lambda_1 = \sum_{j=2}^{4} \beta_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta_4\omega^3 = 1.\omega + 1.\omega^2 + 1.\omega^3 = \omega + \omega^2 + \omega^3,$

$$\omega = e^{\frac{2\pi i}{4}} = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = i, \qquad \omega^2 = e^{\frac{4\pi i}{4}} = \cos\pi + i\sin\pi = -1,$$

$$\omega^3 = e^{\frac{6\pi i}{4}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

$\lambda_1 = i - 1 - i = -1.$
For V=2

$\lambda_2 = \sum_{j=2}^{4} a_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 = 1.\omega^2 + 1.\omega^4 + 1.\omega^6 = \omega^2 + \omega^4 + \omega^6,$

$$\omega^2 = e^{\frac{4\pi i}{4}} = \cos\pi + i\sin\pi = -1, \qquad \omega^4 = e^{\frac{8\pi i}{4}} = \cos 2\pi + i\sin 2\pi = 1,$$

$$\omega^6 = e^{\frac{12\pi i}{4}} = \cos 3\pi + i\sin 3\pi = -1$$

$\lambda_2 = -1 + 1 - 1 = -1.$
For V=3

$\lambda_3 = \sum_{j=2}^{4} \beta_j \omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + \beta_4\omega^9 = 1.\omega^3 + 1.\omega^6 + 1.\omega^9 = \omega^3 + \omega^6 + \omega^9,$

$$\omega^3 = e^{\frac{2\pi i}{4}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i, \qquad \omega^6 = e^{\frac{12\pi i}{4}} = \cos 3\pi + i\sin 3\pi = -1,$$

$$\omega^9 = e^{\frac{18\pi i}{4}} = \cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2} = -i$$

$\lambda_3 = -i - 1 + i = -1.$
For V=4

$\lambda_4 = \sum_{j=2}^{4} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} = 1.\omega^4 + 1.\omega^8 + 1.\omega^{12} = \omega^4 + \omega^8 + \omega^{12},$

$$\omega^4 = e^{\frac{8\pi i}{4}} = \cos 2\pi + i\sin 2\pi = 1, \qquad \omega^8 = e^{\frac{16\pi i}{4}} = \cos 4\pi + i\sin 4\pi = 1,$$

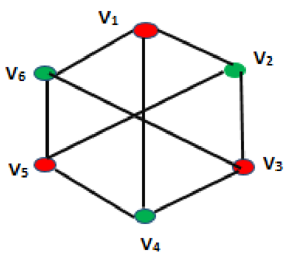
$$\omega^{12} = e^{\frac{24\pi i}{4}} = \cos 6\pi + i\sin 6\pi = 1$$

$\lambda_4 = 1 + 1 + 1 = 3.$

Thus, $\mathrm{spect}(G) = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$.

Regular Graph with 5 Vertices
Each vertex's degree must be 3 for a graph with 5 vertices to be 3-regular. As a result, the total number of degrees must be 5 * 3 = 15.We know that on a graph, the sum of the degrees must be even. Hence a 3-regular graph with 5 vertices is impossible.

Regular Graph with 6 Vertices
The adjacency matrix of the 3-Regular graph with 6 vertices is as follows:



$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Determine eigenvalues of a circulant matrix:
Suppose n=6 and V=1, 2, 3, 4, 5, 6, we can find the eigenvalues using the equation (1).
For V=1

$$\lambda_1 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta_4\omega^3 + \beta_5\omega^4 + \beta_6\omega^5$$

$$= 1.\omega + 0.\omega^2 + 1.\omega^3 + 0.\omega^4 + 1.\omega^5 = \omega + \omega^3 + \omega^5,$$

$$\omega = e^{\frac{2\pi i}{6}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \omega^3 = e^{\frac{6\pi i}{6}} = \cos\pi + i\sin\pi = -1,$$

$$\omega^5 = e^{\frac{10\pi i}{6}} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2},$$

$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2} - 1 + \frac{1}{2} - \frac{i\sqrt{3}}{2} = 0.$

For V=2

$\lambda_2 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 + \beta_5\omega^8 + \beta_6\omega^{10}$

$=1.\omega^2 + 0.\omega^4 + 1.\omega^6 + 0.\omega^8 + 1.\omega^{10} = \omega^2 + \omega^6 + \omega^{10},$

$\omega^2 = e^{\frac{4\pi i}{6}} = cos\frac{2\pi}{3} + isin\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \omega^6 = e^{\frac{12\pi i}{6}} = cos2\pi + isin2\pi = 1,$

$\omega^{10} = e^{\frac{20\pi i}{6}} = cos\frac{10\pi}{3} + isin\frac{10\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2},$

$\lambda_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} + 1 - \frac{1}{2} - \frac{i\sqrt{3}}{2} = 0.$

For V=3

$\lambda_3 = \sum_{j=2}^{6} \beta\omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + \beta_4\omega^9 + \beta_5\omega^{12} + \beta_6\omega^{15}$

$=1.\omega^3 + 0.\omega^6 + 1.\omega^9 + 0.\omega^{12} + 1.\omega^{15} = \omega^3 + \omega^9 + \omega^{15},$

$\omega^3 = e^{\frac{6\pi i}{6}} = cos2\pi + isin2\pi = -1, \quad \omega^9 = e^{\frac{18\pi i}{6}} = cos3\pi + isin3\pi = -1,$

$\omega^{15} = e^{\frac{30\pi i}{6}} = cos5\pi + isin5\pi = -1,$

$\lambda_3 = -1 - 1 - 1 = -3.$

For V=4

$\lambda_4 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} + \beta_5\omega^{16} + \beta_6\omega^{20}$

$=1.\omega^4 + 0.\omega^8 + 1.\omega^{12} + 0.\omega^{16} + 1.\omega^{20} = \omega^4 + \omega^{12} + \omega^{20},$

$\omega^4 = e^{\frac{8\pi i}{6}} = cos\frac{4\pi}{3} + isin\frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad \omega^{12} = e^{\frac{24\pi i}{6}} = cos4\pi + isin4\pi = 1,$

$\omega^{20} = e^{\frac{40\pi i}{6}} = cos\frac{20\pi}{3} + isin\frac{20\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2},$

$\lambda_4 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} + 1 - \frac{1}{2} + \frac{i\sqrt{3}}{2} = 0.$

For V=5

$\lambda_5 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^5 + \beta_3\omega^{10} + \beta_4\omega^{15} + \beta_5\omega^{20} + \beta_6\omega^{25}$

$=1.\omega^5 + 0.\omega^{10} + 1.\omega^{15} + 0.\omega^{20} + 1.\omega^{25} = \omega^5 + \omega^{15} + \omega^{25},$

$\omega^5 = e^{\frac{10\pi i}{6}} = cos\frac{5\pi}{3} + isin\frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad \omega^{15} = e^{\frac{30\pi i}{6}} = cos5\pi + isin5\pi = -1,$

$\omega^{25} = e^{\frac{50\pi i}{6}} = cos\frac{25\pi}{3} + isin\frac{25\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2},$

$\lambda_5 = \frac{1}{2} - \frac{i\sqrt{3}}{2} - 1 + \frac{1}{2} + \frac{i\sqrt{3}}{2} = 0.$

For V=6

$\lambda_6 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^6 + \beta_3\omega^{12} + \beta_4\omega^{18} + \beta_5\omega^{24} + \beta a_6\omega^{30}$

$=1.\omega^6 + 0.\omega^{12} + 1.\omega^{18} + 0.\omega^{24} + 1.\omega^{30} = \omega^6 + \omega^{18} + \omega^{30},$

$\omega^6 = e^{\frac{12\pi i}{6}} = cos2\pi + isin2\pi = 1, \quad \omega^{18} = e^{\frac{36\pi i}{6}} = cos6\pi + isin6\pi = 1,$

$\omega^{30} = e^{\frac{60\pi i}{6}} = cos10\pi + isin10\pi = 1,$

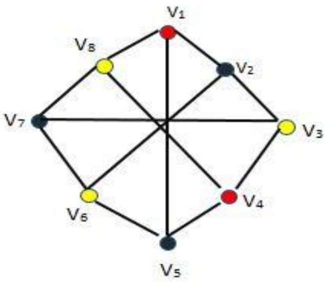
$\lambda_6 = 1 + 1 + 1 = 3.$

Thus, $spect(G) = \begin{bmatrix} -3 & 0 & 3 \\ 1(-3) & 4(0) & 1(3) \end{bmatrix}.$

Regular Graph with 7 Vertices

Each vertex's degree must be 3 for a graph with 5 vertices to be 3-regular. As a result, the total number of degrees must be 7 * 3 = 21.We know that on a graph, the sum of the degrees must be even. Hence a 3-regular graph with 5 vertices is impossible.

Regular Graph with 8 Vertices
The adjacency matrix of the 3-Regular graph with 8 vertices is as follows:



$$\begin{bmatrix} 0&1&0&0&1&0&0&1\\ 1&0&1&0&0&1&0&0\\ 0&1&0&1&0&0&1&0\\ 0&0&1&0&1&0&0&1\\ 1&0&0&1&0&1&0&0\\ 0&1&0&0&1&0&1&0\\ 0&0&1&0&0&1&0&1\\ 1&0&0&1&0&0&1&0 \end{bmatrix}$$

Determine eigenvalues of a circulant matrix:
Suppose n=8 and V=1, 2, 3, 4, 5, 6, 7, 8, we can find the eigenvalues using the equation (1).
For V=1

$$\lambda_1 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta_4\omega^3 + \beta_5\omega^4 + \beta_6\omega^5 + \beta_7\omega^6 + \beta_8\omega^7$$
$$=1.\omega + 0.\omega^2 + 0.\omega^3 + 1.\omega^4 + 0.\omega^5 + 0.\omega^6 + 1.\omega^7 = \omega + \omega^4 + \omega^7,$$
$$\omega = e^{\frac{2\pi i}{8}} = cos\frac{\pi}{4} + isin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}, \qquad \omega^4 = e^{\frac{8\pi i}{8}} = cos\pi + isin\pi = -1,$$
$$\omega^7 = e^{\frac{14\pi i}{8}} = cos\frac{7\pi}{4} + isin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2},$$
$$\lambda_1 = 0.414.$$

For V=2

$$\lambda_2 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 + \beta_5\omega^8 + \beta_6\omega^{10} + \beta_7\omega^{12} + \beta_8\omega^{14}$$
$$=1.\omega^2 + 0.\omega^4 + 0.\omega^6 + 1.\omega^8 + 0.\omega^{10} + 0.\omega^{12} + 1.\omega^{14} = \omega^2 + \omega^8 + \omega^{14},$$
$$\omega^2 = e^{\frac{4\pi i}{8}} = cos\frac{\pi}{2} + isin\frac{\pi}{2} = i, \qquad \omega^8 = e^{\frac{16\pi i}{8}} = cos2\pi + isin2\pi = 1,$$
$$\omega^{14} = e^{\frac{28\pi i}{8}} = cos\frac{7\pi}{2} + isin\frac{7\pi}{2} = -i,$$
$$\lambda_2 = 1.$$

For V=3

$$\lambda_3 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + +\beta_4\omega^9 + \beta_5\omega^{12} + \beta_6\omega^{15} + \beta_7\omega^{18} + \beta_8\omega^{21}$$
$$=1.\omega^3 + 0.\omega^6 + 0.\omega^9 + 1.\omega^{12} + 0.\omega^{15} + 0.\omega^{18} + 1.\omega^{21} = \omega^3 + \omega^{12} + \omega^{21},$$
$$\omega^3 = e^{\frac{6\pi i}{8}} = cos\frac{3\pi}{4} + isin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}, \qquad \omega^{12} = e^{\frac{24\pi i}{8}} = cos3\pi + isin3\pi = -1,$$
$$\omega^{21} = e^{\frac{42\pi i}{8}} = cos\frac{21\pi}{4} + isin\frac{21\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2},$$
$$\lambda_3 = -2.414.$$

For V=4

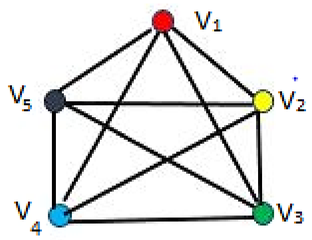
$$\lambda_4 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} + \beta_5\omega^{16} + \beta_6\omega^{20} + a_7\omega^{24} + \beta_8\omega^{28}$$
$$=1.\omega^4 + 0.\omega^8 + 0.\omega^{12} + 1.\omega^{16} + 0.\omega^{20} + 0.\omega^{24} + 1.\omega^{28} = \omega^4 + \omega^{16} + \omega^{28},$$
$$\omega^4 = e^{\frac{8\pi i}{8}} = cos\pi + isin\pi = -1, \qquad \omega^{16} = e^{\frac{32\pi i}{8}} = cos4\pi + isin4\pi = 1,$$
$$\omega^{28} = e^{\frac{56\pi i}{8}} = cos7\pi + isin7\pi = -1,$$
$$\lambda_4 = -1.$$

For V=5

$$\lambda_5 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^5 + \beta_3\omega^{10} + \beta_4\omega^{15} + \beta_5\omega^{20} + \beta_6\omega^{25} + \beta_7\omega^{30} + \beta_8\omega^{35}$$
$$=1.\omega^5 + 0.\omega^{10} + 0.\omega^{15} + 1.\omega^{20} + 0.\omega^{25} + 0.\omega^{30} + 1.\omega^{35} = \omega^5 + \omega^{20} + \omega^{35},$$
$$\omega^5 = e^{\frac{10\pi i}{8}} = cos\frac{5\pi}{4} + isin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}, \qquad \omega^{20} = e^{\frac{40\pi i}{8}} = cos5\pi + isin5\pi = -1,$$

$\omega^{35} = e^{\frac{70\pi i}{8}} = cos\frac{35\pi}{4} + isin\frac{35\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2},$
$\lambda_5 = -2.414.$
For V=6
$\lambda_6 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^6 + \beta_3\omega^{12} + \beta_4\omega^{18} + \beta_5\omega^{24} + \beta_6\omega^{32} + \beta_7\omega^{38} + \beta_8\omega^{42}$
$=1.\omega^6 + 0.\omega^{12} + 0.\omega^{18} + 1.\omega^{24} + 0.\omega^{32} + 0.\omega^{38} + 1.\omega^{42} = \omega^6 + \omega^{24} + \omega^{42},$
$\omega^6 = e^{\frac{12\pi i}{8}} = cos\frac{6\pi}{4} + isin\frac{6\pi}{4} = - - i, \quad \omega^{24} = e^{\frac{48\pi i}{8}} = cos6\pi + isin6\pi = 1,$
$\omega^{42} = e^{\frac{84\pi i}{8}} = cos\frac{21\pi}{2} + isin\frac{21\pi}{2} = i,$
$\lambda_6 = 1.$
For V=7
$\lambda_7 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^7 + \beta_3\omega^{14} + \beta_4\omega^{21} + \beta_5\omega^{28} + \beta_6\omega^{35} + \beta_7\omega^{42} + \beta_8\omega^{49}$
$=1.\omega^7 + 0.\omega^{14} + 0.\omega^{21} + 1.\omega^{28} + 0.\omega^{35} + 0.\omega^{42} + 1.\omega^{49} = \omega^7 + \omega^{28} + \omega^{49},$
$\omega^7 = \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}, \quad \omega^{28} = -1, \quad \omega^{49} = e^{\frac{98\pi i}{8}} = \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2},$
$\lambda_7 = 0.414.$
For V=8
$\lambda_8 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^8 + \beta a_3\omega^{16} + \beta_4\omega^{24} + \beta_5\omega^{32} + \beta_6\omega^{40} + \beta_7\omega^{48} + \beta_8\omega^{56}$
$=1.\omega^8 + 0.\omega^{16} + 0.\omega^{24} + 1.\omega^{32} + 0.\omega^{40} + 0.\omega^{48} + 1.\omega^{56} = \omega^8 + \omega^{32} + \omega^{56},$
$\omega^8 = 1, \quad \omega^{32} = 1, \quad \omega^{56} = 1,$
$\lambda_8 = 3.$
Thus, $spect(G) = \begin{bmatrix} -2.414 & -1 & 0.414 & 1 & 3 \\ 2 & 1 & 2 & 2 & 1 \end{bmatrix}.$

Regular Graph with 5 Vertices
The adjacency matrix of the 4-Regular graph with 5 vertices is as follows:

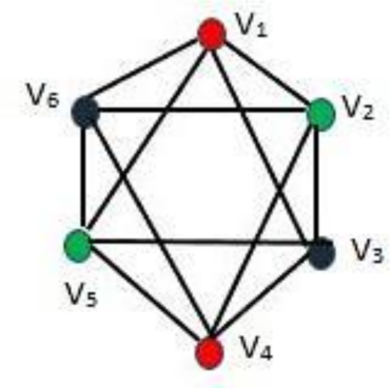


$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Determine eigenvalues of a circulant matrix:
Suppose n=5 and V=1, 2, 3, 4, 5, we can find the eigenvalues using the equation (1).
For V=1
$\lambda_1 = \sum_{j=2}^{5} \beta_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta_4\omega^3 + \beta_5\omega^4$
$=1.\omega + 1.\omega^2 + 1.\omega^3 + 1.\omega^4 = \omega + \omega^2 + \omega^3 + \omega^4 = -1.$
For V=2
$\lambda_2 = \sum_{j=2}^{5} \beta_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 + \beta_5\omega^8$
$=1.\omega^2 + 1.\omega^4 + 1.\omega^6 + 1.\omega^8 = \omega^2 + \omega^4 + \omega^6 + \omega^8 = -1.$
For V=3
$\lambda_3 = \sum_{j=2}^{5} \beta_j \omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + \beta_4\omega^9 + \beta_5\omega^{12}$
$=1.\omega^3 + 1.\omega^6 + 1.\omega^9 + 1.\omega^{12} = \omega^3 + \omega^6 + \omega^9 + \omega^{12} = -1.$
For V=4
$\lambda_4 = \sum_{j=2}^{5} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} + \beta_5\omega^{16}$
$=1.\omega^4 + 1.\omega^8 + 1.\omega^{12} + 1.\omega^{16} = \omega^4 + \omega^8 + \omega^{12} + \omega^{16} = -1.$
For V=5
$\lambda_5 = \sum_{j=2}^{5} \beta_j \omega^{(j-1)V} = \beta_2\omega^5 + \beta_3\omega^{10} + \beta_4\omega^{15} + \beta_5\omega^{20}$
$=1.\omega^5 + 1.\omega^{10} + 1.\omega^{15} + 1.\omega^{20} = \omega^5 + \omega^{10} + \omega^{15} + \omega^{20} = 4.$
Thus, $spect(G) = \begin{bmatrix} -1 & 4 \\ 4 & 1 \end{bmatrix}.$

Regular Graph with 6 Vertices

The adjacency matrix of the 4-Regular graph with 6 vertices is as follows:



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Determine eigenvalues of a circulant matrix:

Suppose n=6 and V=1, 2, 3, 4, 5, 6, we can find the eigenvalues using the equation (1).

For V=1

$\lambda_1 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta a_4\omega^3 + \beta_5\omega^4 + \beta_6\omega^5$

$=1.\omega + 1.\omega^2 + 0.\omega^3 + 1.\omega^4 + 1.\omega^5 = \omega + \omega^2 + \omega^4 + \omega^5 = 0.$

For V=2

$\lambda_2 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 + \beta_5\omega^8 + \beta_6\omega^5$

$=1.\omega^2 + 1.\omega^4 + 0.\omega^6 + 1.\omega^8 + 1.\omega^{10} = \omega^2 + \omega^4 + \omega^8 + \omega^{10} = -2.$

For V=3

$\lambda_3 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + \beta_4\omega^9 + \beta_5\omega^{12} + \beta_6\omega^{15}$

$=1.\omega^3 + 1.\omega^6 + 0.\omega^9 + 1.\omega^{12} + 1.\omega^{15} = \omega^3 + \omega^6 + \omega^{12} + \omega^{15} = 0.$

For V=4

$\lambda_4 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} + \beta_5\omega^{16} + \beta_6\omega^{20}$

$=1.\omega^4 + 1.\omega^8 + 0.\omega^{12} + 1.\omega^{16} + 1.\omega^{20} = \omega^4 + \omega^8 + \omega^{16} + \omega^{20} = -2.$

For V=5

$\lambda_5 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^5 + \beta_3\omega^{10} + \beta_4\omega^{15} + \beta_5\omega^{20} + \beta_6\omega^{25}$

$=1.\omega^5 + 1.\omega^{10} + 0.\omega^{15} + 1.\omega^{20} + 1.\omega^{25} = \omega^5 + \omega^{10} + \omega^{20} + \omega^{25} = 0.$

For V=6

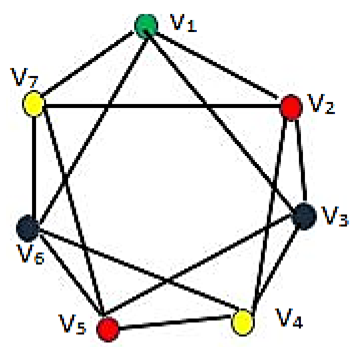
$\lambda_6 = \sum_{j=2}^{6} \beta_j \omega^{(j-1)V} = \beta_2\omega^6 + \beta_3\omega^{12} + \beta_4\omega^{18} + \beta_5\omega^{24} + \beta_6\omega^{30}$

$=1.\omega^6 + 1.\omega^{12} + 0.\omega^{18} + 1.\omega^{24} + 1.\omega^{30} = \omega^6 + \omega^{12} + \omega^{24} + \omega^{30} = 4.$

Thus, $spect(G) = \begin{bmatrix} -2 & 0 & 4 \\ 2 & 3 & 1 \end{bmatrix}$.

Regular Graph with 7 Vertices

The adjacency matrix of the 4-Regular graph with 7 vertices is as follows:



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

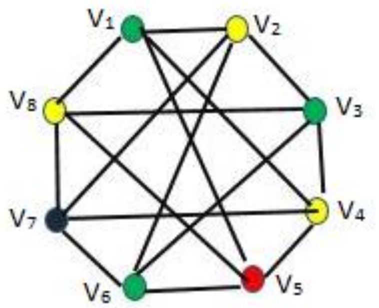
Determine eigenvalues of a circulant matrix:

Suppose n=7 and V=1, 2, 3, 4, 5, 6, 7, we can find the eigenvalues using the equation (1).

For V=1

$\lambda_1 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta_4\omega^3 + \beta_5\omega^4 + \beta_6\omega^5 + \beta_7\omega^6$
$=1.\omega + 1.\omega^2 + 0.\omega^3 + 0.\omega^4 + 1.\omega^5 + 1.\omega^6 = \omega + \omega^2 + \omega^5 + \omega^6 = 0.8.$
For V=2
$\lambda_2 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 + \beta_5\omega^8 + \beta_6\omega^{10} + \beta_7\omega^{12}$
$=1.\omega^2 + 1.\omega^4 + 0.\omega^6 + 0.\omega^8 + 1.\omega^{10} + 1.\omega^{12} = \omega^2 + \omega^4 + \omega^{10} + \omega^{12} = -2.25.$
For V=3
$\lambda_3 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + \beta_4\omega^9 + \beta_5\omega^{12} + \beta_6\omega^{15} + \beta_6\omega^{18}$
$=1.\omega^3 + 1.\omega^6 + 0.\omega^9 + 0.\omega^{12} + 1.\omega^{15} + 1.\omega^{18} = \omega^3 + \omega^6 + \omega^{15} + \omega^{18} = -0.55.$
For V=4
$\lambda_4 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} + \beta_5\omega^{16} + \beta_6\omega^{20} + \beta_7\omega^{24}$
$=1.\omega^4 + 1.\omega^8 + 0.\omega^{12} + 0.\omega^{16} + 1.\omega^{20} + 1.\omega^{24} = \omega^4 + \omega^8 + \omega^{20} + \omega^{24} = -0.55.$
For V=5
$\lambda_5 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega^5 + \beta_3\omega^{10} + \beta_4\omega^{15} + \beta_5\omega^{20} + \beta_6\omega^{25} + \beta_7\omega^{30}$
$=1.\omega^5 + 1.\omega^{10} + 0.\omega^{15} + 0.\omega^{20} + 1.\omega^{25} + 1.\omega^{30} = \omega^5 + \omega^{10} + \omega^{25} + \omega^{30} = -2.25.$
For V=6
$\lambda_6 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega^6 + \beta_3\omega^{12} + \beta_4\omega^{18} + \beta_5\omega^{24} + \beta a_6\omega^{30} + \beta_7\omega^{36}$
$=1.\omega^6 + 1.\omega^{12} + 0.\omega^{18} + 0.\omega^{24} + 1.\omega^{30} + 1.\omega^{36} = \omega^6 + \omega^{12} + \omega^{30} + \omega^{36} = 0.80.$
For V=7
$\lambda_7 = \sum_{j=2}^{7} \beta_j \omega^{(j-1)V} = \beta_2\omega^7 + \beta_3\omega^{14} + \beta_4\omega^{21} + \beta_5\omega^{28} + \beta_6\omega^{35} + \beta_7\omega^{42}$
$=1.\omega^7 + 1.\omega^{14} + 0.\omega^{21} + 0.\omega^{28} + 1.\omega^{35} + 1.\omega^{42} = \omega^7 + \omega^{14} + \omega^{35} + \omega^{42} = 4.$
Thus, $spect(G) = \begin{bmatrix} -2.25 & -0.55 & 0.8 & 4 \\ 2 & 2 & 2 & 1 \end{bmatrix}$.

Regular Graph with 8 Vertices
The adjacency matrix of the 4-Regular graph with 8 vertices is as follows:



$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Determine eigenvalues of a circulant matrix:
Suppose n=8 and V=1, 2, 3, 4, 5, 6, 7, 8, we can find the eigenvalues using the equation (1).
For V=1
$\lambda_1 = \sum_{j=2}^{8} a_j \omega^{(j-1)V} = \beta_2\omega + \beta_3\omega^2 + \beta_4\omega^3 + \beta_5\omega^4 + \beta_6\omega^5 + \beta_7\omega^6 + \beta_8\omega^7$
$= 1.\omega + 0.\omega^2 + 1.\omega^3 + 1.\omega^4 + 0.\omega^5 + 0.\omega^6 + 1.\omega^7$
$= \omega + \omega^3 + \omega^4 + \omega^7 = -1 + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$
For V=2
$\lambda_2 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^2 + \beta_3\omega^4 + \beta_4\omega^6 + \beta_5\omega^8 + \beta\omega^{10} + \beta_7\omega^{12} + a_8\omega^{14}$
$=1.\omega^2 + 0.\omega^4 + 1.\omega^6 + 1.\omega^8 + 0.\omega^{10} + 0.\omega^{12} + a_8\omega^{14}$
$= \omega^2 + \omega^6 + \omega^8 + \omega^{14} = 1 - i.$
For V=3
$\lambda_3 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^3 + \beta_3\omega^6 + \beta_4\omega^9 + \beta_5\omega^{12} + \beta_6\omega^{15} + \beta_6\omega^{18} + \beta_7\omega^{21}$
$=1.\omega^3 + 0.\omega^6 + 1.\omega^9 + 1.\omega^{12} + 0.\omega^{15} + 0.\omega^{18} + 1.\omega^{21}$
$= \omega^3 + \omega^9 + \omega^{12} + \omega^{21} = -1 - \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$
For V=4
$\lambda_4 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^4 + \beta_3\omega^8 + \beta_4\omega^{12} + \beta a_5\omega^{16} + \beta_6\omega^{20} + \beta_7\omega^{24} + \beta_8\omega^{28}$
$=1.\omega^4 + 0.\omega^8 + 1.\omega^{12} + 1.\omega^{16} + 0.\omega^{20} + 0.\omega^{24} + 1.\omega^{28}$
$= \omega^4 + \omega^{12} + \omega^{16} + \omega^{28} = -2.$

For V=5

$\lambda_5 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^5 + \beta_3\omega^{10} + \beta_4\omega^{15} + \beta_5\omega^{20} + \beta_6\omega^{25} + \beta_7\omega^{30} + \beta_7\omega^{35}$

$=1.\omega^5 + 0.\omega^{10} + 1.\omega^{15} + 1.\omega^{20} + 0.\omega^{25} + 0.\omega^{30} + 1.\omega^{35}$

$= \omega^5 + \omega^{15} + \omega^{20} + \omega^{35} = -1 - \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$

For V=6

$\lambda_6 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^6 + \beta_3\omega^{12} + \beta_4\omega^{18} + \beta_5\omega^{24} + \beta_6\omega^{30} + \beta_7\omega^{36} + \beta_8\omega^{42}$

$=1.\omega^6 + 0.\omega^{12} + 1.\omega^{18} + 1.\omega^{24} + 0.\omega^{30} + 0.\omega^{36} + 1.\omega^{42}$

$= \omega^6 + \omega^{18} + \omega^{24} + \omega^{42} = 1 + i.$

For V=7

$\lambda_7 = \sum_{j=2}^{8} \beta_j \omega^{(j-1)V} = \beta_2\omega^7 + \beta_3\omega^{14} + \beta_4\omega^{21} + \beta_5\omega^{28} + \beta_6\omega^{35} + \beta_7\omega^{42} + \beta_8\omega^{49}$

$=1.\omega^7 + 0.\omega^{14} + 1.\omega^{21} + 1.\omega^{28} + 0.\omega^{35} + 0.\omega^{42} + 1.\omega^{49}$

$= \omega^7 + \omega^{21} + \omega^{28} + \omega^{49} = -1 + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$

For V=8

$\lambda_8 = \sum_{j=2}^{8} a_j \omega^{(j-1)V} = \beta_2\omega^8 + \beta_3\omega^{16} + \beta_4\omega^{24} + \beta_5\omega^{32} + \beta_6\omega^{40} + \beta_7\omega^{48} + \beta_8\omega^{56}$

$=1.\omega^8 + 0.\omega^{16} + 1.\omega^{24} + 1.\omega^{32} + 0.\omega^{40} + 0.\omega^{48} + 1.\omega^{56}$

$= \omega^8 + \omega^{24} + \omega^{32} + \omega^{56} = 4.$

Thus,

$$spect(G) = \begin{bmatrix} 1+\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} & 1-i & -1-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} & -2 & -\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2} & 1+i & -1+\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2} & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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TABLE 1

The Relationship between 3-Regular Graph and Eigenvalues

No. of Vertex	Rank of Matrix	Chromatic Index	No. of clique	Eigenvalues
4	4	4	3	(-1) (-1) (-1) **(3)**
6	2	2	3	(-3) (0) (0) (0) (0) **(3)**
8	4	3	3	(−2.414) (−2.414) (−1) (0.414) (0.414)(1) (1) **(3)**

TABLE II

The Relationship between 4-Regular Graph and Eigenvalues

No. of Vertex	Rank of Matrix	Chromatic Index	No. of clique	Eigenvalues
5	5	4	4	(-1) (-1) (-1) (-1) **(4)**
6	3	3	3	(-2) (-2) (0) (0) (0) **(4)**
7	7	4	3	(-2.25) (-2.25) (-0.55) (-0.55) (0.8) (0.8) **(4)**
8	4	4	3	$(1+\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})(1-i)(-1-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})(-2)(-1-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})(1+i)(-1+\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})$**(4)**

3. References

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