Improved Approach for Oscillatory Solutions of Second Order Half-Linear Neutral Delay Difference Equation

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Abstract. The second-order half-linear neutral delay difference equation obtains some improved oscillatory solutions in this research. $\Delta\left(\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}\right)+\gamma(\mu)\beta^{\delta}(\mu-\sigma)=0, \qquad \mu\geq\mu_0,$

where$\delta>0$is an odd positive integer quotient. Our findings in this work are crisp and improved on some of the literature's well-known oscillatory results. Some examples are provided to highlight our primary findings.**2020 Mathematics Subject Classifications:** 39A12, 39A21.

1. INTRODUCTION

The oscillation and non-oscillation of solutions of differential equations has piqued researchers' curiosity in recent years. The oscillation of the second order half-linear neutral delay difference equation is the focus of this research.

$$\Delta\left(\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}\right)+\gamma(\mu)\beta^{\delta}(\mu-\sigma)=0, \qquad \mu\geq\mu_0, \quad (1.1)$$

whereΔsignifies the forward difference operator $\Delta\beta(\mu)=\beta(\mu+1)-\beta(\mu)$for any sequence $\{\beta(\mu)\}$of real numbers. We will assume the following throughout this paper: σ is a fixed nonnegative integer,$\delta>0$ is the of odd positivequotient integers, $\{\alpha(\mu)\}_{\mu=\mu_0}^{\infty}$and $\{\gamma(\mu)\}_{\mu=\mu_0}^{\infty}$are both fixed nonnegative integers. are real-number sequences in which$\alpha(\mu)>0, \gamma(\mu)>0$ and $\{\gamma(\mu)\}_{\mu=\mu_0}^{\infty}$each have a positive subsequence, and

$$\sum_{\mu=\mu_0}^{\infty}\left(\frac{1}{\alpha(\mu)}\right)^{\frac{1}{\delta}}=\infty, \qquad (1.2)$$

$$\sum_{\mu=\mu_0}^{\infty}\left(\frac{1}{\alpha(\mu)}\right)^{\frac{1}{\delta}}<\infty, \qquad (1.3)$$

A nontrivial sequence$\{\beta(\mu)\}$ that is defined for $\mu\geq-\sigma$ and satisfies equation (1.1) for $\mu=0,1,2,3\cdots$is referred to as a solution of (1.1). Obviously, if

$$\beta(\mu)=X(\mu) for \mu=-\sigma,\dots,-1,\mu_0-1, \qquad (1.4)$$

If the initial parameters are established, then (1.1) has a unique solution that satisfies (1.4). A solution$\{\beta(\mu)\}$of (1.1) is considered to be oscillatory if there exists an$\mu \geq \mu_1$ such that $\beta(\mu), \beta(\mu)+1 \leq 0$ for every $\mu_1 > \mu_0$. Otherwise, it's referred to as nonoscillatory. Equations that are half-linear and have the form as

$$\Delta\left(\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}\right) + \gamma(\mu+1)a^{\delta}(\mu+1) = 0, \qquad \mu = 0,1,2,3\ldots, \qquad (1.5)$$

The studies of Agarwal [2,6], Zhang [5], Dinakar et al. [7], Thandapani et al. [3,4], Gopalakrishnan [8], [15] and Murugesan [9], [14]have received a great deal of attention in the literature in recent years. In several of the articles cited above, Riccati equations of various forms are used to derive oscillatory solutions that ensure that every nontrivial solution is oscillatory or nonoscillatory.

We create some unique oscillatory solutions and Kamanev-type oscillation conditions for using Riccati transformation techniques (1.1). In Section 2, we examine at the scenario where (1.2) and (1.3) holds and develop certain necessary and sufficient conditions for all solutions of to oscillate (1.1). In Section 3, we construct certain necessary conditions that ensure that any solution $\{\beta(\mu)\}$of (1.1) oscillates. Our results extend and enhanceSaker [13]andThandapani et al. [10], [11]. The major findings are illustrated with few examples.

2. MAIN RESULTS OF OSCILLATION

In this section, we examine at the condition where (1.2) holds and define the necessary criteria for all solutions to oscillate (1.1). First, we consider all possible solutions (1.1).

Theorem 2.1. Let us consider $\delta > 0$ and$\Delta\alpha(\mu) \geq 0$ be an eventually positive solution of (1.1). If every solution of the neutral delay difference equation.

$$\Delta\varphi(\mu) + \frac{\gamma(\mu)}{\alpha(\mu-\sigma)}\left(\frac{\mu-\sigma}{2}\right)^{\delta}\varphi(\mu-\sigma) = 0, \quad \mu \geq \mu_1 \geq \mu_0 (2.1)$$

oscillates, without loss of generality every solution of (1.1) oscillates for all $\delta > 0$.

Proof. Let us assume$\{\beta(\mu)\}$is a finally positive solution to the problem (1.1).
such that $\beta(\mu) > 0$and $\beta(\mu-\sigma) > 0$for all $\mu \geq \mu_1 \geq \mu_0$. Therefore $\varphi(\mu) = -\beta(\mu)$transforms(1.1). From (1.1) we have

$$\Delta\left(\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}\right) = -\gamma(\mu)\beta^{\delta}(\mu-\sigma) = 0, \qquad \mu \geq \mu_1, \qquad (2.2)$$

and so $\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}$ is an eventually nonincreasing sequence. Clearly,$\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}$is eventually positive. Indeed, since$\{\gamma(\mu)\}_{\mu=\mu_0}^{\infty}$ has a positive subsequence and the non-decreasing sequence $\{\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}\}$is either eventually positive. If an integer exists$\mu_2 \geq \mu_1$such that $\alpha(\mu_2)\left(\Delta\beta(\mu_2)\right)^{\delta} = k < 0$for $\mu \geq \mu_2$, then (2.2) implies that $\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta} \leq \alpha(\mu_2)\left(\Delta\beta(\mu_2)\right)^{\delta} = k$, hence

$$\Delta\beta(\mu) \leq (k)^{\frac{1}{\delta}}\left(\frac{1}{\alpha(\mu)}\right)^{\frac{1}{\delta}},$$

Such that

$$\beta(\mu) \leq \beta(\mu_2) + (k)^{\frac{1}{\delta}}\sum_{i=\mu_2}^{\mu-1}\left(\frac{1}{\alpha_i}\right)^{\frac{1}{\delta}} \to -\infty as \mu \to \infty (2.3)$$

Which is a contradictionof$\beta(\mu) > 0$. Hence $\alpha(\mu)\left(\Delta\beta(\mu)\right)^{\delta}$is eventually positive. Therefore, we see that there is some $\mu_1 \geq \mu_0$such that.

$$\beta(\mu) > 0, \Delta\,\beta(\mu) \geq 0, \Delta\left(\alpha(\mu)\left(\Delta\,\beta(\mu)\right)^{\delta}\right) \leq 0, \qquad \mu \geq \mu_1, \qquad (2.4)$$

From (2.4), since $\Delta\left(\alpha(\mu)\left(\Delta\,\beta(\mu)\right)^{\delta}\right) \leq 0$, then we have $\Delta^2\beta(\mu) \leq 0$ for $\mu \geq \mu_0$. If not there exists $\mu_2 \geq \mu_1$such that $\Delta^2\beta(\mu) > 0$and this implies that$\Delta\,\beta(\mu+1) > \Delta\,\beta(\mu)$, so that since $\Delta\alpha(\mu) \geq 0$, $\alpha(\mu+1)\left(\Delta\,\beta(\mu+1)\right)^{\delta} > \alpha(\mu+1)\left(\Delta\,\beta(\mu)\right)^{\delta} \geq \alpha(\mu)\left(\Delta\,\beta(\mu)\right)^{\delta}$andthis contradicts the fact that $\{\alpha(\mu)\left(\Delta\,\beta(\mu)\right)^{\delta}\}$is nonincreasing sequence. Then $\Delta^2\beta(\mu) \leq 0$,whichimplies that $\beta(\mu) - \beta(\mu_1) = \sum_{c=\mu_1}^{\mu-1}\Delta\beta(c) \geq (\mu-\mu_1)\,\Delta\,\beta(\mu)$which leads to$\beta(\mu) \geq \frac{\mu}{2}\Delta\,\beta(\mu)$ for $\mu \geq \mu_2 \geq 2\mu_1 + 1$. Such that,

$$\beta(\mu-\sigma) \geq \frac{\mu-\sigma}{2}\Delta\beta(\mu-\sigma), \;\; \mu \geq \mu_3 = \mu_2 + \sigma \tag{2.5}$$

Hence, from (2.5) and (1.1), we have

$$\Delta\left(\alpha(\mu)\left(\Delta\,\beta(\mu)\right)^{\delta}\right) + \gamma(\mu)\left(\frac{\mu-\sigma}{2}\right)^{\delta}\left(\Delta\beta(\mu-\sigma)\right)^{\delta} \leq 0, \;\; \mu \geq \mu_3 \tag{2.6}$$

Set $\varphi(\mu) = \alpha(\mu)\left(\Delta\,\beta(\mu)\right)^{\delta}$, then $\varphi(\mu) > 0$ and satisfies

$$\Delta\varphi(\mu) + \frac{\gamma(\mu)}{\alpha(\mu-\sigma)}\left(\frac{\mu-\sigma}{2}\right)^{\delta}\varphi(\mu-\sigma) \leq 0, \quad \mu \geq \mu_3 (2.7)$$

As a result, the neutral advance delay difference equation (2.1) has an eventually positive solution, which is the contradiction of the assumption that (2.1) has an oscillating solution. Then every (1.1) solution is oscillatory.

Corollary 2.1. Assume that (1.2) holds. Additionally, if there exists a positive sequence $\{\omega(\mu)\}_{\mu=0}^{\infty}$such that for all $\rho \geq 1$,

$$\lim_{n\to\infty}\sup\frac{1}{n^{\rho}}\sum_{?=0}^{n-1}(n-\mu)^{\rho}\;[\,\omega(\mu)\gamma(\mu) - \frac{\omega(\mu+1)^2}{4\overline{\omega(\mu)}}K_{n,\mu}] = \infty \tag{2.8}$$

Then without loss of generality every solution of (1.1) oscillates for all $\delta \geq 1$.

3. SOME APPLICATIONS OF OSCILLATORY SOLUTIONS

Let us assume that (1.3) holds, and let $\{\omega(\mu)\}$ be an eventually positive sequence. Then every solution of (1.1) oscillates or$\log_{\mu\to\infty}\beta(\mu) = 0$. The primary outcomes in this section are illustrated in the following examples.

Example 3.1. Consider the second order half-linear neutral delay difference equation

$$\Delta((\mu+1)^2\Delta\beta(\mu) + \psi\beta(\mu-1) = 0, \quad \mu \geq 1, \tag{3.1}$$

where $\psi > 1/4$. Then, $\alpha(\mu) = (\mu+1)^2$, $\delta = 1$and $\sigma = 1$. If we take$\omega(\mu) = \mu$, then one can easily see that (3.1) holds, and

$$\sum_{z=\mu_0}^{\mu}\left[Z\gamma(z) - \frac{\alpha(z-1)((Z-1)-(z))^2}{4z}\right] = \sum_{z=1}^{\mu}\left[\psi z - \frac{z^2}{4z}\right] = \sum_{z=1}^{\mu}\left[\frac{(4\psi-1)z}{4}\right] \to \infty$$

as $\mu \to \infty$. Thusevery solution of (3.1) oscillates or $\beta(\mu) \to\; 0$as $\mu \to \infty$.

Example 3.2. Consider the following second order half-linear neutral advance delay difference equation

$$\Delta^2(\beta(\mu-1) + \frac{1}{\sqrt{\mu}}\beta(\mu-2) + \frac{\psi}{\mu^2}\beta(\mu-1) = 0, \quad \mu \geq 1, \tag{3.2}$$

$\psi > 0$is a constant.

$$\lim_{n\to\infty} \sup \frac{1}{n^\rho} \sum_{\mu=0}^{n-1} (n-\mu)^\rho \left[\omega(\mu) - \frac{(\mu+1)^2}{4\overline{Z(\mu)}}\right] = \infty \tag{3.3}$$

where

$$\omega(\mu) = \frac{2+(4\psi-1)\mu}{4\mu(\mu+1)} - \frac{\psi}{\sqrt{(\mu+1)^3}}$$

which implies

$$\lim_{n\to\infty} \sup \frac{1}{n^2} \sum_{\square=0}^{n-1} \square \left[(n-\mu)^2 \left(\frac{2+(4\psi-1)\mu}{4\mu(\mu+1)} - \frac{\psi}{\sqrt{(\mu+1)^3}}\right) - \frac{(\mu+1)^2}{\lambda+1}\right] = \infty \tag{3.4}$$

If $\psi > 1/4$, then every solution of equation (3.4) is oscillatory.

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