ZERO DIVISOR GRAPH REVELATION AND SO ITS MULTIFARIOUS SCOPE

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ABSTRACT:

The zero-divisor graph of a non-commutative ring R can be described as the directed graph ℾ(R) whose vertices are all non-zero zero-divisors of R and in which, for any two different vertices x and y, $x \rightarrow y$ is an edge if and only if xy=0. We look at how R's ring-theoretic and graph-theoretic aspects ℾ(R) interact. In this work, it is demonstrated that, with a finite number of exceptions, if R is a ring and S is a finite semisimple ring that is not a field and, ℾ(R)≅ ℾ(S) then R≅ S. We display that if R is a ring and ℾ(R)≅ ℾ(M□(F)), then R≅ M□(F). By putting off all instructions from the edges in Redmond's definition of the easy undirected design ℾ(R). We categorise any ring R whose $\overline{\Gamma}$(R) as both a whole graph, a bipartite graph, and a tree.

Keywords: Zero-divisor, Non-commutative ring, Directed graph, Matrix ring.

1 | INTRODUCTION

R will always denote a commutative ring with the character intended by 1. The set Z(R) will show how the zero-divisors are arranged, and Z ∗(R) = Z(R){ 0} shows how the non-zero-divisors of R are arranged. U represents the configuration of all the units in a ring R. (R). We denote the ring of entire figures by Zn. n modulo The zero-divisor map of a commutative ring R with 1 is denoted by (R). It is a simple undirected example whose vertex set is Z ∗(R) , and the two vertices u, v ∈ Z ∗(R) are adjacent if and provided that uv = vu = 0 [1]. The thought of zero-divisor plan of a commutative ring used to be first delivered with the aid of Beck [3] and in his work all the factors of a ring R had been the vertices of the plan and two vertices x and y have been adjoining if and solely if x y = zero. A extraordinary strategy of associating a design to a commutative ring R was once given via Anderson and Livingston in [4], the place the design Γ (R) has its vertices as factors of Z∗(R) = Z(R)\{0} and two vertices x, y ∈ Z∗(R) are adjoining if and solely if x y = zero The authors believed that this definition higher illustrates the zero-divisor shape of the ring [5].

This plan turns out to best exhibit the homes of the set of zero-divisors and different associated homes of a commutative ring. The zero-divisor design interprets the algebraic residences of a ring to format theoretical tools, as a consequence helps in exploring fascinating effects in each layout concept and summary algebra [6]. A diagram G with vertex set V(G) = ∅ and part set E(G) of unordered pairs of distinct vertices is referred to as a easy graph. The cardinality of V(G) is known as the order of G and the cardinality of E(G) is referred to as its size. A layout G is linked if and solely if there exists a pat between each pair of vertices u and v [7]. A plan on n vertices such that any pair of distinct vertices is joined by means of an part is known as a whole graph, denoted via Kn. A entire subgraph of G of biggest order is known as a maximal clique of G and its order is referred to as the clique number of G, denoted through cl(G).

The wide variety of edges incident on a vertex is referred to as its diploma and a vertex of diploma 1 is known as a pendent vertex. The biggest diploma of a vertex is denoted by Δ and smallest diploma is denoted by using δ. In a linked design G, the distance between two vertices u and v is the size of the shortest route between u and v. The diameter of a design G is described as diam(G) = sup{(d(u, v) | u, v ∈ V(G))}, the place d(u, v) denotes the distance between vertices u and v of G. The possibility of a zero-divisor graph was once brought through I. Beck [8] in 1988, and afterward correspondingly concentrated by utilizing D. D. Anderson and M. Naseer [8]. Notwithstanding, they let every one of the elements of R be vertices of the diagram, and they have been frequently associated with colorings. Our meaning of Γ (R) and the accentuation on the cooperation between the diagram hypothetical homes of Γ (R) and the ring-hypothetical homes of R are expected to D. F. Anderson and P. S. Livingston [9] in 1999.

The beginnings and early records of zero-divisor diagrams will be referenced in additional component in Area 7. The 2d region begins off evolved with the paper [14] that checked the amazing measure of shape existing in Γ (R). It used to be this shape that pulled in ring scholars to the spot in the expectations that the diagram hypothetical shape might need to unveil basic algebraic shape in Z(R). The ensuing various segments main focus on some imperative chart hypothesis impacts in regards to Γ (R). Planar and toroidal zero-divisor diagrams are completely portrayed in Segment 6 [10]. The end part offers a short records of

Γ (R) underscoring the true inquiries that animated the area and notices a few speculations of Γ (R). Most confirmations are overlooked in the leisure activity of curtness, and we do never again announce to give all important impacts in this field.

The catalogue is our endeavour at providing an impressive posting of distributions around here, however large numbers of the papers are currently not unequivocally expressed in this review. We ensuing review a few standards from format hypothesis. Allow G to be a (undirected) diagram. We say that G is connected assuming there is a course between any two superb vertices. For unmistakable vertices x and y in G, the distance among x and y, meant with the guide of d(x,y), is the size of a briefest course interfacing x and y (d(x,x) = zero and d(x,y) = ∞ if no such course exits). The breadth of G is diam(G) = sup{d(x,y) | x and y are vertices of G}. A pattern of size n in G is a course of the construction x1 −x2 −···−xn −x1, where xi = xj when I = j. We frame the circumference of G, signified by utilizing gr(G), as the size of a briefest cycle in G, provided G incorporates a cycle; in any case, gr(G) = ∞. At last, a vertex of G is a stop assuming it is abutting to exactly one different vertex. An arrangement G is entire in the event that any two wonderful vertices are nearby.

The total diagram with n vertices will be signified by utilizing Kn (we empower n to be an innumerable cardinal). An entire bipartite design is a configuration G which could likewise be divided into two disjoint nonempty vertex units An and B with the end goal that two great vertices are connecting if and provided that they are in wonderful vertex sets. On the off chance that one of the vertex units is a singleton, we name G a celebrity diagram. We mean the entire bipartite plan through Km,n, where |A| = m and |B| = n (once more, we grant m and n to be incalculable cardinals); so a star chart is a K1,n. All the more by and large, G is entire r-partite assuming G is the disjoint association of r nonempty vertex units and two magnificent vertices are connecting if and exclusively assuming they are in great vertex sets. At last, let Km,3 be the plan molded through turning into an individual from G1 = Km,3 (= A∪B with |A| = m and |B| = 3) to the superstar design G2 = K1,m with the guide of sorting out the focal point of G2 and a component of B [11].

A subgraph G of a chart G is an achieved subgraph of G in the event that two vertices of G are contiguous in G if and exclusively assuming they are abutting in G. Obviously, gr(G) ≥ gr(G) when G is a provoked subgraph of G, but there is no connection between diam(G) what's more, diam(G). An entire subgraph of G is alluded to as an inner circle. The inner circle scope of G, signified by utilizing cl(G), is the greatest whole number r ≥ 1 to such an extent that Kr ⊆ G (if Kr ⊆ G for all numbers r ≥ 1, then we compose cl(G) = ∞). The chromatic assortment of G, denoted by χ(G), is the negligible amount of tints wished to conceal the vertices of G so that no two bordering vertices have the equivalent tone. Obviously cl(G) ≤ χ(G) [12].

Underneath we outfit a few instances of zero-divisor diagrams. We will never again recognize between isomorphic charts (two diagrams G and G are isomorphic on the off chance that there is a bijection f between the vertices of G and the vertices of G to such an extent that x and y are nearby in G if and exclusively if f(x) and f(y) are bordering in G) [13]. To the surprise of no one, Z, Zn, Q, R, C, furthermore, Fq will indicate the numbers, numbers modulo n, sane numbers, genuine numbers, complex numbers, and the limited region with q components, separately. In Segment 5, circles will now and again be conveyed to vertices of Γ (R) comparing to zero-divisors x with x2 = 0.

2 | RELATED WORK

In this paper it is shown that for any finite commutative ring *R*, the edge chromatic number of *Γ*(*R*) is equal to the maximum degree of *Γ*(*R*), unless *Γ*(*R*) is a complete graph of odd order. In this article we explore the relationship between $\Gamma_I(R) \cong \Gamma_J(S)$ and Γ(*R*/*I*) ≅ Γ(*S*/*J*). We also discuss when $\Gamma_I(R)$ is bipartite [15]. Finally we give some results on the subgraphs and the parameters of $\Gamma_I(R)$. More precisely, we prove that if *R* is a local ring with at least 33 elements, and *Γ*(*R*)≠∅, then *Γ*(*R*) is not planar. We use the set of the associated primes to find the minimal length of a cycle in *Γ*(*R*). Also, we determine the rings whose zero-divisor graphs are complete *r*-partite graphs and show that for any ring *R* and prime number *p*, *p*⩾3, if *Γ*(*R*) is a finite complete *p*-partite graph, then $|Z(R)|=p^2$, $|R|=p^3$, and *R* is isomorphic to exactly one of the rings Zp3, Zp[x,y](xy,y2−x), Zp2[y](py,y2−ps), where 1⩽*s*<*p*.

In a manner analogous to the commutative case, the zero-divisor graph of a non-commutative ring *R* can be defined as the directed graph T(R) that its vertices are all non-zero zero-divisors of *R* in which for any two distinct vertices *x* and *y*, x→y is an edge if and only if xy=0 [15]. This article studies the zero divisor graph for the ring of Gaussian integers modulo *n*, Γ ($\mathbb{Z}_n[i]$). For each positive integer *n*, the number of vertices, the diameter, the girth and the case when the dominating number is 1 or 2 is found. For a commutative ring R with unity (1 6= 0), the zero-divisor graph of R, denoted by Γ(R), is a simple graph with vertices as elements of R and two distinct vertices are adjacent whenever the product of the vertices is zero. This article aims at gaining a deeper insight into the basic structural properties of zero-divisor graphs given by Beck [16].

This article surveys the recent and active area of zero-divisor graphs of commutative rings. Notable algebraic and graphical results are given, followed by a historical overview and an extensive bibliography. An algorithm is presented for constructing the zero-divisor graph of a direct product of integral domains. Moreover, graphs which are realizable as zero-divisor graphs of direct products of integral domains are classified, as well as those of Boolean rings [17]. In particular, graphs which are realizable as zero-divisor graphs of finite reduced commutative rings are classified. Let R be a commutative ring with nonzero identity and $Z(R)$ its set of zero-divisors. The zero-divisor graph of R is $\Gamma(R)$, with vertices $Z(R)\backslash\{0\}$ and distinct vertices x and y are adjacent if and only if $xy = 0$. For a proper ideal I of R, the ideal-based zero-divisor graph of R is $\Gamma_I(R)$, with vertices $\{x \in R\backslash I \mid xy \in I$ for some $y \in R\backslash I\}$ and distinct vertices x and y are adjacent if and only if $xy \in I$ *[18]*.

In this article, we study the relationship between the two graphs $\Gamma(R)$ and $\Gamma_I(R)$. We also determine when $\Gamma_I(R)$ is either a complete graph or a complete bipartite graph and investigate when $\Gamma_I(R) \cong \Gamma(S)$ for some commutative ring S *[19]*. In this paper, we associate the graph $\Gamma_I(N)$ to an ideal I of a near-ring N. We exhibit some properties and structure of $\Gamma_I(N)$. For a commutative ring R, Beck conjectured that both chromatic number and clique number of the zero-divisor graph Γ(R) of R are equal. We prove that Beck's conjecture is true for $\Gamma_I(N)$. Moreover, we characterize all right permutable near-rings N for which the graph $\Gamma_I(N)$ is finitely colorable. It is shown that, for a fixed positive integer g, there are finitely many isomorphism classes of rings whose zero-divisor graph has genus g *[20]*.

The proof can then be modified to yield an analogous result for nonorientable genus. . Let R be a commutative ring and let Γ(Zn) be the zero divisor graph of a commutative ring R, whose vertices are non-zero zero divisors of Zn, and such that the two vertices u, v are adjacent if n divides uv [21]. In this paper, we introduce the concept of Decomposition of Zero Divisor Graph in a commutative ring and also discuss some special cases of Γ(Z2 2p), Γ(Z3 2p), Γ(Z5 2p), Γ(Z7 2p) and Γ(Zp2q). In this paper we answer this question in the affirmative. We prove that if R is any local ring with more than 27 elements, and R is not a field, $\Gamma_I(R)$ then is not planar [22]. Moreover, we determine all finite commutative local rings whose zero-divisor graph is planar. Let R be a commutative ring and Γ (R) denote its zero-divisor graph. In this paper we investigate the crosscap number of the non-orientable compact surface which Γ (R) can be embedded and illustrate all finite commutative rings R (up to isomorphism) such that Γ (R) is projective [23-25].

3 | DIAMETER, GIRTH AND GIRTH OF Γ (R(+)M)

We discuss expressions of Γ (R) and when (R) has measurement 2 or circumference 4. (R). The authors confirmed that Γ (R) and Γ (T (R)) are isomorphic as illustrated in [2, Hypothesis 2.2]. Particularly, the width and size of Γ(R) and Γ(T (R)) are same.

Theorem 2.1. ([43, Theorem 1], [14, Theorem 2.2]) Let R be a commutative ring.Then Γ (R) is finite if and solely if both R is finite or R is an necessary domain. Inparticular, if $1 \leq |\Gamma (R)|$ < ∞, then R is finite and now not a field. Moreover, $|R| \leq |Z(R)|$

2 if R is now not an critical domain.

Proof. It is enough to show the "moreover" statement. Let $x \in Z(R)*$. Then the R-module homomorphism $f : R \longrightarrow R$ given through $f(r) = rx$ has kernel annR(x) and image xR. Thus $|R| = |annR(x)||xR| \leq |Z(R)|$

2. The first "big" end result in [14] confirmed that Γ (R) is usually linked and relatively "compact."

Theorem 2.2. ([14, Theorem 2.3]) Let R be a commutative ring. Then Γ (R) is connected with $diam(\Gamma (R)) \leq 3$.

Proof. Let $x,y \in Z(R)*$ be distinct. We will show that $d(x,y) \leq$ three If $xy = 0$, then $d(x,y) = 1$. So feel that xy is nonzero. There are $z,w \in Z(R)*$ such that $xz = wy =$ zero If $zw = 0$, then $x - zw - y$ is a route of size 2; so $d(x,y) =$ two If $zw = 0$, then $x-z-w-y$ is a route of size at most three (we should have $x = z$ or $w = y$). Thus, $d(x,y) \leq 3$, and for this reason Γ (R) is related and $diam(\Gamma (R)) \leq 3$.

If G consists of a cycle, then $gr(G) \leq 2 \cdot diam(G) + 1$ [38, Proposition 1.3.2]. So, if Γ (R) includes a cycle, then $gr(\Gamma (R)) \leq 7$ with the aid of Theorem 2.2. Anderson and Livingston, however, seen that all of the examples they regarded had girths of 3,4, or ∞. Based on this, they conjectured that if a zero-divisor diagram has a cycle, then its girth is three or four They have been in a position to show this if the ring used to be Artinian (e.g., finite) [14, Theorem 2.4]. The conjecture was once tested independently by way of S. B. Mulay [64] and F.

DeMeyer and K. Schneider [36]. Additionally, brief proofs have been given by means of M. Axtell, J. Coykendall, and J. Stickles [17] and S. Wright [84].

Theorem 2.3. ([14, Theorem 2.4], [64, (1.4)], [36, Theorem 1.6]) Let R be a commutative ring. If Γ (R) incorporates a cycle, then gr(Γ (R)) ≤ 4.

Proof. Assume with the aid of way of contradiction that n = gr(Γ(R)) is 5,6, or 7. Let x1 − x2 −···− xn − x1 be a cycle of minimal length. So, x1x3 = zero If x1x3 = xi for 1 ≤ i ≤ n, then x2 −x3 −x4 −x1x3 −x2 is a 4-cycle, a contradiction. Thus, x1x3 = xi for some 1 ≤ i ≤ n. If x1x3 = x1, then x1 −x2 −x3 −x4 −x1 is a 4-cycle. If x1x3 = x2, then x2 − x3 − x4 − x2 is a 3-cycle. If x1x3 = xn, then x1 − x2 − xn − x1 is a 3-cycle. Hence, x1x3 = x1,x2, cycle in Γ (R), and gr(Γ (R)) ≤ 4.

Thus, diam(Γ (R)) ∈ {0,1,2,3} and gr(Γ (R)) ∈ {3,4,∞}. The examples given in the Introduction exhibit that all these viable values may also occur. The subsequent result expands on Theorem 2.3.

Theorem 2.4. Let R be a commutative ring which is now not an essential domain. Then exactly one of the following holds:

(a) Γ (R) has a cycle of length three or four (i.e., gr(Γ (R)) ≤ 4);

(b) Γ (R) is a singleton or a famous person graph; or

(c) Γ (R) = K1,3 (i.e., R ∼= Z2 ×Z4 or R ∼= Z2 ×Z2[X]/(X2)).

Moreover, if Γ (R) carries a cycle, then each vertex of Γ (R) is both an stop or part of a 3-cycle or a 4-cycle.

Proof. The finite case used to be located in [14, p. 349], whilst the popular case is independently given in [36, Theorem 1.6] and [64, (1.4), (2.0), and (2.1)]. The “moreover” announcement is from [64, (1.4) and (2.1)]. Another characterization of girth was once given in [15] the use of the truth that R and T(R) have isomorphic zero-divisor graphs (Theorem 4.4). The following two theorems explicitly signify when the girth of a zero-divisor design is four or ∞, and thus implicitly when the girth is 3.

Theorem 2.5. ([15, Theorems 2.2 and 2.4]) Let R be a decreased commutative ring.

(a) The following statements are equivalent.

(1) gr(Γ (R)) = 4.

(2) T(R) = K1 ×K2, the place every Ki is a area with |Ki| ≥ 3.

(3) Γ (R) = Km,n with m,n ≥ 2.

(b) The following statements are equivalent.

(1) Γ (R) is nonempty with gr(Γ (R)) = ∞.

(2) T(R) = Z2 ×K, the place K is a field.

(3) Γ (R) = K1,n for some n ≥ 1.

Theorem 2.6. ([15, Theorems 2.3 and 2.5]) Let R be a commutative ring with nil(R) nonzero.

(a) The following statements are equivalent.

(1) gr(Γ (R)) = 4.

(2) R ∼= D × B, the place D is an essential area with |D| ≥ three and B = Z4 or Z2[X]/(X2). (Thus T(R) ∼= T(D)×B.)

(3) Γ (R) = Km,3 with m ≥ 2.

(b) The following statements are equivalent.

(1) gr(Γ (R)) = ∞.

(2) R ∼= B or R ∼= Z2 × B, the place B = Z4 or Z2[X]/(X2), or Γ (R) is a star graph

(3) Γ (R) is a singleton, a K1,3 , or a K1,n for some n ≥ 1.

Much of the lookup on zero-divisor graphs has centered on the girth and diameter for positive instructions of rings. For example, gr(Γ (R)) is studied in phrases of the number of related high beliefs of R in [3], and homes of Γ (R) for a reduced ring R are associated to topological residences of Spec(R) in [74]. The girth and diameter of the zero-divisor sketch of the direct product of two commutative rings (not necessarily with identity) are characterised in [21], and for diameter these thoughts are extended to finite direct merchandise in [41]. Also, the girth and diameter of the zerodivisor layout of an idealization are characterised in [18] and [15], and the girth and diameter of Γ (R → I) (the amalgamated duplication of a ring R alongside an perfect I [33]) are studied in [62]. The girth and diameter of Γ (R) for a commutative ring R which satisfies positive divisibility prerequisites on factors or comparability prerequisites on ideals or high beliefs are investigated in [10].

We subsequent supply a extra specified dialogue of the zero-divisor graphs for polynomial rings and electricity sequence rings. First, we think about the less complicated case for girth.

Theorem 2.7. ([17, Theorem 4.3], [15, Theorem 3.2]) Let R be a commutative ring.

(a) Suppose that Γ (R) is nonempty with gr(Γ (R)) = ∞.

(1) If R is reduced, then gr(Γ (R[X])) = gr(Γ (R[[X]])) = 4.

(2) If R is no longer reduced, then gr(Γ (R[X])) = gr(Γ (R[[X]])) = 3.

(b) If gr(Γ (R)) = 3, then gr(Γ (R[X])) = gr(Γ (R[[X]])) = 3.

(c) Suppose that gr(Γ (R)) = 4.

(1) If R is reduced, then gr(Γ (R[X])) = gr(Γ (R[[X]])) = 4.

(2) If R is now not reduced, then gr(Γ (R[X])) = gr(Γ (R[[X]])) = 3.

Proof. From [17, Theorem 4.3], we have gr(Γ(R)) ≤ gr(Γ(R[X])) = gr(Γ (R[[X]])), and equality holds if R is decreased and Γ (R) includes a cycle. The last cases and the end result as cited above are from [15, Theorem 3.2].

The "diameter" case is no longer so easy. This was once first studied in [17], and some cases for non-Noetherian commutative rings left open in [17] had been resolved by means of T. G. Lucas in [59]. However, we are content material right here to simply point out the reduced case; the interested reader must refer to [17, 59], and [15] for associated results. In particular, see [59] Theorems 3.4 and 3.6] for polynomial rings and [59, Section 5] for electricity series rings. Recall that a ring R is a McCoy ring if every finitely generated best contained in Z(R) has a nonzero annihilator.

Theorem 2.8. ([59, Theorem 4.9]) Let R be a decreased commutative ring that is not an indispensable domain. Then

$1 \leq \mathrm{diam}(\Gamma (R)) \leq \mathrm{diam}(\Gamma (R[X])) \leq \mathrm{diam}(\Gamma (R[[X]])) \leq 3.$

Moreover, right here are all viable sequences for these dimensions.

(1) diam(Γ(R)) = 1 and diam(Γ (R[X])) = diam(Γ (R[[X]])) = two if and solely if R ∼= Z2 ×Z2.

(2) diam(Γ (R)) = diam(Γ (R[X])) = diam(Γ (R[[X]])) = two if and solely if either R has precisely two minimal primes and is now not isomorphic to Z2 ×Z2 or for every pair of countably generated beliefs I and J with nonzero annihilators, the sum I +J has a nonzero annihilator (and R is a McCoy ring with Z(R) an ideal).

(3) diam(Γ(R)) = diam(Γ (R[X])) = two and diam(Γ (R[[X]])) = three if and solely if R is a McCoy ring with Z(R) an perfect however there exists countably generated beliefs I and J with nonzero annihilators such that I +J does now not have a nonzero annihilator.

(4) diam(Γ(R)) = two and diam(Γ (R[X])) = diam(Γ (R[[X]])) = three if and solely if Z(R) is an perfect and every two generated perfect contained in Z(R) has a nonzero annihilator however R is no longer a McCoy ring.

(5) diam(Γ (R)) = diam(Γ (R[X])) = diam(Γ (R[[X]])) = three if and solely if R has more than two minimal primes and there is a pair of zero-divisors a and b such that (a,b) does now not have a nonzero annihilator.

Let $A \subseteq B$ be an extension of commutative rings with identity. In this case, Γ (A) is an precipitated subgraph of Γ (B). It may additionally show up that Γ (A) = Γ (B) for $A \subset B$ (this occurs if and solely if A is a pullback of a finite nearby ring [12, Theorem 4.3]). It is clear that $\mathrm{gr}(\Gamma (B)) \leq \mathrm{gr}(\Gamma (A))$. Moreover, for all $m,n \in \{3,4,\infty\}$ with $m \leq n$, there is a suitable extension $A \subset B$ of decreased finite commutative rings such that gr(Γ (B)) = m and gr(Γ (A)) = n [9, Example 2.1]. Again, the case for the diameter is no longer so clear due to the fact though Z(A) ⊆ Z(B), it want now not be the case that Z(A) = Z(B) ∩ A. In fact, for m,n ∈ {0,1,2,3}, there is a suitable extension A ⊂ B of commutative rings with diam(Γ (A)) = m and diam(Γ(B)) = n unless (m,n) ∈ {(0,0),(1,0),(2,0),(2,1),(3,0),(3,1)} [9, Proposition 3.2]. Thus,

diam(Γ (A)) ≤ diam(Γ (B)) until diam(Γ (A)) = three and diam(Γ (B)) = 2; specific examples with diam(Γ (A)) = three and diam(Γ(B)) = two are given in [18, Example 3.7] and [9, Example 3.7]. The subsequent theorem offers stipulations when this can happen.

Theorem 2.9. (a) ([9, Theorem 3.8]) Let A be a commutative ring with diam(Γ(A)) = three Then there is a commutative extension ring B of A such that diam(Γ (B)) = 2 if and solely if Z(A) ⊆ M for some maximal best M of A. Moreover, if A is reduced, then B can additionally be chosen to be reduced.

(b) ([9, Corollary 3.12]) Let A ⊆ B be an extension of commutative rings with dim(A) = zero Then diam(Γ (A)) ≤ diam(Γ (B)). In particular, this holds if A is Artinian or a finite commutative ring Part (b) surely follows from section (a) given that diam(Γ (R)) ≤ two when Z(R) = nil(R) [9, Lemma 3.11]. Theorem 2.9 illustrates a case the place the zero-divisor graph of an endless ring can also behave alternatively otherwise from that of a finite ring. Also note that if B is an overring of A, then diam(Γ (A)) = diam(Γ(B)) via Corollary 4.5(a).

The above consequences reveal that the zero-divisor format of a commutative ring exhibits a magnificent quantity of graphical structure that may want to possibly grant some insight into the algebraic shape of Z(R). The subsequent numerous sections exhibit some of the outcomes in which Γ (R) affords records about R and Z(R).

Lemma 2.1. Let R be a commutative ring with complete quotient ring T (R). Then diam(Γ(T (R))) = diam(Γ(R)) and gr(Γ(T (R))) = gr(Γ(R)).

Proof. Let T = T (R). Clearly diam(Γ(T)) = 1 if and solely if diam(Γ(R)) = 1. Suppose that diam(Γ(T)) = 2.

Then diam(Γ(R)) ≥ two Let a, b ∈ Z(R) ∗ with a 6= b and ab 6= zero Then aq = zero = bq for some q ∈ Z(T) ∗ − {a, b}.

Let $q = c/t$ with $c \in R$ and $t \in R - Z(R)$. Then ac = zero = bc. Thus d(a, b) = 2, and for this reason diam(Γ(R)) = two A similar argument indicates that diam(Γ(T)) = two if diam(Γ(R)) = two The end result for the diameter now follows considering the diameter of a zero-divisor diagram is at most three [3, Theorem 2.3]. Since Γ(R) is a subgraph of Γ(T), without a doubt gr(Γ(T)) ≤ gr(Γ(R)). Suppose that gr(Γ(T)) = three Then there are distinct nonzero factors $q_1, q_2, q_3 \in T$ such that $q_1q_2 = q_2q_3 = q_3q_1$ = zero Let every $q_i = a_i /t$ with $a_i \in R$ and $t \in R - Z(R)$. Then a_1, a_2, a_3 are awesome factors in R with $a_1a_2 = a_2a_3 = a_3a_1$ = zero Thus $a_1 - a_2 - a_3 - a_1$ is a triangle in Γ(R); so gr(Γ(R)) = three Similarly, gr(Γ(R)) = four if gr(Γ(T)) = four The end result for the girth now follows since the girth of a zero-divisor sketch is either 3, 4, or ∞ [13, (1.4)].

Following [11], we say that wonderful vertices a and b in a diagram G are orthogonal, written a⊥b, if a and b are adjacent and there is no vertex c which is adjoining to each a and b, i.e., the side a − b is no longer phase of any triangle of G. As in [2], we say that G is complemented if for every vertex a of G, there is a vertex b of G such that a⊥b, and that G is uniquely complemented if G is complemented and every time a⊥b and a⊥c, then b and c are adjoining to precisely the same vertices. In [2], the authors categorized the commutative rings R such that Γ(R) is complemented or uniquely complemented. For example, a decreased commutative ring R is complemented if and solely if it is uniquely complemented, if and solely if T (R) is von Neumann ordinary [2, Theorem 3.5]. Note that if gr(Γ(R)) = four or ∞ (with |Γ(R)| ≥ 2), then Γ(R) is complemented. However, if $R = Z_2 \times Z_2 \times Z_2$, then Γ(R) is (uniquely) complemented and gr(Γ(R)) = 3.

We subsequent use the above principles and consequences from [2] to decide when gr(Γ(R)) = four We have two cases, depending on whether or not or no longer R has any nonzero nilpotent elements.

Theorem 2.2. The following statements are equal for a decreased commutative ring R.

(1) gr(Γ(R)) = 4.

(2) $T(R) = K_1 \times K_2$, the place every K_i is a subject with $|K_i| \geq 3$.

(3) $\Gamma(R) = K_{m,n}$ with $m, n \geq 2$.

Proof. (1) ⇒ (2) Suppose that gr(Γ(R)) = four Then Γ(R) is complemented. Thus T = T (R) is von Neumann regular by [2, Theorem 3.5] and no longer a field. Hence T has a nontrivial idempotent, and hence T = T1×T2. Suppose that there are 0 6= x, y ∈ T1 with x y = zero (note that x 6= y when you consider that R, and as a result T , is reduced). Then (x, 0)−(y, 0)−(0, 1)−(x, 0) is a triangle in Γ(T), a contradiction due to the fact gr(Γ(T)) = gr(Γ(R)) = four via Lemma 2.1. Thus T1 is an indispensable domain, in fact, a field. Similarly, T2 ought to additionally be a field. Hence T = K1 × K2 for fields K1 and K2. If both K1 or K2 has only 2 elements, then Γ(T) is a famous person graph. In this case, gr(Γ(T)) = ∞, a contradiction considering the fact that gr(Γ(T)) = gr(Γ(R)) = 4

by Lemma 2.1.

(2) ⇒ (3) This follows on account that the graphs Γ(R) and Γ(T) are isomorphic [2, Theorem 2.2] and Γ(K1×K2) = K m,n , where m = |K1| − 1 and n = |K2| − 1.

(3) ⇒ (1) This is clear.

Theorem 2.3. The following statements are equal for a commutative ring R with nil(R) nonzero.

(1) gr(Γ(R)) = 4.

(2) R ∼= D× B, the place D is an crucial area with |D| ≥ three and B = Z4 or Z2[X]/(X2). (Thus T (R) ∼= T (D)× B.)

(3) Γ(R) = K m,3 with m ≥ 2.

Proof. (1) ⇒ (2) Suppose that gr(Γ(R)) = four Then Γ(R) is complemented. If Γ(R) is uniquely complemented, then Γ(R) is a famous person layout [2, Theorem 3.9], and consequently gr(Γ(R)) = ∞, a contradiction. Thus R ∼= D × B, the place D is an integral area and B = Z4 or Z2[X]/(X2) through [2, Theorem 3.14]. Hence Γ(R) = Km,3 , the place m = |D| − 1. We must have |D| ≥ three due to the fact that in any other case gr(Γ(R)) = ∞, a contradiction.

The implications (2) ⇒ (3) and (3) ⇒ (1) are each clear.

Next we decide when gr(Γ(R)) = ∞ the usage of thoughts from [2]. Similar outcomes have additionally been received in [8, Theorems 1.7 and 1.12] and [13] (cf. Remark 2.6). Again, we have two cases, relying on whether or not or now not R is reduced. Since gr(Γ(R)) = 3, four or ∞, we have thus, in some sense, additionally characterised when gr(Γ(R)) = 3.

Theorem 2.4. The following statements are equal for a decreased commutative ring R.

(1) Γ(R) is nonempty with gr(Γ(R)) = ∞.

(2) T (R) = Z2 × K , the place K is a field.

(3) Γ(R) = K 1,n for some n ≥ 1.

Proof. (1) ⇒ (2) Suppose that gr(Γ(R)) = ∞ and Γ(R) 6= ∅. Then |Γ(R)| ≥ two considering that R is reduced, and for that reason Γ(R) is complemented. As in the proof of (1) ⇒ (2) of Theorem 2.2, we have T (R) = K1 × K2 for fields K1 and K2. If each discipline has at least three elements, then gr(Γ(R)) = four by way of Theorem 2.2, a contradiction. Hence we might also expect that K1 has two elements; so K1 = Z2.

(2) ⇒ (3) This follows considering the fact that the graphs Γ(R) and Γ(T (R)) are isomorphic [2, Theorem 2.2] and Γ(Z2 × K) =K1,n , the place n = |K| − 1.

(3) ⇒ (1) This is clear.

Theorem 2.5. The following statements are equal for a commutative ring R with nil(R) nonzero.

(1) gr(Γ(R)) = ∞.

(2) R ∼= B or R ∼= Z2 × B, the place B = Z4 or Z2[X]/(X2), or Γ(R) is a superstar graph.

(3) Γ(R) is a singleton, a K 1,3 , or a K1,n for some n ≥ 1.

Proof. (1) ⇒ (2) Suppose that gr(Γ(R)) = ∞. If Γ(R) is a point, then R ∼= Z4 or Z2[X]/(X 2). So count on that

$\Gamma(R)$ has at least two elements. Then $\Gamma(R)$ is complemented. If $\Gamma(R)$ is uniquely complemented, then $\Gamma(R)$ is a star graph by using [2, Theorem 3.9]. If $\Gamma(R)$ is no longer uniquely complemented, then $R \sim= D \times B$, the place D is an vital domain

and $B = Z4$ or $Z2[X]/(X2)$ through [2, Theorem 3.14]. If $|D| \geq 3$, then $gr(\Gamma(R))$ = four as in Theorem 2.3, a contradiction.

Thus $|D| = 2$; so $D = Z2$.

The implications (2) ⇒ (3) and (3) ⇒ (1) are each clear.

Theorem 2.6. Let R be a commutative ring with $diam(\Gamma(R)) \leq$ two Then precisely one of the following holds.

(1) Z(R) is an (prime) perfect of R.

(2) $T(R) = K1 \times K2$, the place every Ki is a field.

Proof. Let $T = T(R)$. Note that (1) holds if and solely if T has a special maximal ideal. So consider that $diam(\Gamma(R)) \leq 2$ and that Z(R) is no longer a high perfect of R. Then there are wonderful maximal beliefs M and N of T . Thus $x + y = 1$ for some $x \in M$ and $y \in N$, and consequently $ann(x) \cap ann(y) = \{0\}$. Since $diam(\Gamma(T)) = diam(\Gamma(R)) \leq$ two with the aid of Lemma 2.1,we have to have $x y = 0$, and consequently x and y are idempotent. Hence $T = T1 \times T2$. Suppose that there is a $c \in Z(T1)*$.

Then $a = (c, 1)$ and $b = (1, 0)$ are in $Z(T)*$ and $d(a, b) \geq 3$, a contradiction. Thus T1 need to be an indispensable domain, in fact, a field. Similarly, T2 is a field. Hence $T(R) = K1 \times K2$ with every Ki a field. Thus (2) holds.

The following result, a partial communicate to Theorem 2.6, will be beneficial in the subsequent section.

Theorem 2.7. Let R be a (reduced) commutative ring which is no longer an vital area such that R is a subring of $D1 \times D2$, the place every Di is an vital domain. Then both $R \sim= Z2 \times Z2$ (and subsequently $diam(\Gamma(R)) = 1$) or $diam(\Gamma(R)) = 2$.

Proof. If diam(Γ(R)) = 1, then R ∼= Z2 × Z2 [3, Theorem 2.7]. So feel that diam(Γ(R)) ≥ two Let x, y ∈ Z(R) ∗ be awesome with x y 6= zero Then we might also expect that x, y ∈ D1 × {0}. Since x ∈ Z(R) ∗ and R is reduced, there is a z ∈ Z(R) ∗ − {x, y} such that x z = zero But then z ∈ {0} × D2; so x z = yz = 0, and as a result d(x, y) = two Thus diam(Γ(R)) = two

Remark 2.8. (a) We have diam(Γ(R)) = zero if and solely if Γ(R) is a point, i.e., R ∼= Z4 or Z2[X]/(X 2). We have diam(Γ(R)) = 1 if and solely if |Γ(R)| ≥ two and Γ(R) is complete. This takes place if and solely if both R ∼= Z2 × Z2 or Z(R) is a high perfect of R with |Z(R)| ≥ three and Z(R) 2 = {0} [3, Theorem 2.8].

(b) As a speak to Theorem 2.6(2), let T (R) = K1 × K2 be the product of two fields. If K1 = K2 = Z2, then (by Lemma 2.1) diam(Γ(R)) = 1; otherwise, diam(Γ(R)) = 2 As a communicate to phase (1) of Theorem 2.6, be aware that if R is noetherian, then Z(R) is an annihilator best if and solely if it is an (prime) perfect [10, Theorems 6 and 82], and in this case diam(Γ(R)) ≤ two However, it is feasible to have Z(R) an best of a decreased ring R, but diam(Γ(R)) = 3 (see the remarks after [12, Example 5.1]).

(c) A great ideal-theoretic characterization of diam(Γ(R)) is given in [12, Theorem 2.6]. In particular, diam(Γ(R)) = three if and solely if there are awesome a, b ∈ Z(R) ∗ with ann(a) ∩ ann(b) = {0} and both (i) R is reduced with at least three minimal top ideals, or (ii) R is no longer reduced.

GIRTH OF Γ(R(+)M)

When looking at the girth of Γ(R(+)M), matters are very easy if the module is large enough. For if |M| Γ 4, then g(Γ (R(+)M)) = 3, in view that (0, m1) — (0, m2) — (0, m3) – (0, m1) is a cycle of size three (where m1, m2, and m3 are wonderful nonzero factors of M).

So, we solely want to reflect onconsideration on when the module has two or three elements. First we seem to be at when M Γ Z3 and reflect onconsideration on R(+)Z3. In most cases, the girth of Γ (R(+)Z3) is three. One item valuable of observe is that if R has extra than three elements, there constantly exists a nonzero r ∈ R such that r · Z3 = zero To see this, expect r · Z3 Γ = zero for all r ∈ R∗. Then there exist distinct r1, r2 ∈ R∗ such that r1 · 1 = r2 · 1 and subsequently (r1 − r2)1 = zero the place r1 − r2 is nonzero, a contradiction. Also, on the grounds that the module is unitary, the ring can't have fewer than three elements. This is beneficial in our subsequent result.

Theorem 2.1. Let R a commutative ring with identification and M Γ Z3 an R-module. Then

(i) g(Γ (R(+)Z3)) = three if and solely if ann(Z3) Γ = {0}.

(ii) g(Γ (R(+)Z3)) = ∞ if and solely if ann(Z3) = {0}. This happens exactly when R Γ Z3.

Proof.

(i) Assume there exists a nonzero issue r ∈ R such that rZ3 = zero Since (r, 0) − (0, 1) − (0, 2) − (r, 0) is a cycle of size 3, the end result is obvious. The different course is proven through the usage of the contrapositive of the implication confirmed below.

(ii) Assume that rZ3 Γ = zero for each and every nonzero issue r ∈ R. Then r · 1 Γ = zero for allr ∈ R∗. Thus ann((0, 1)) = ann((0, 2)) = {(0, 0), (0, 1), (0, 2)}. Since Γ (R(+)Z3) is connected, we see that R has no nonzero zero divisors; hence, R is an indispensable domain. In lightof the statement preceeding the theorem, R Γ Z3. Since Z(R(+)Z3)∗ = {(0, 1), (0, 2)}, we have g(Γ (R(+)Z3)) = ∞. The different course is established via the use of the contrapositive of the implication tested in (i). Note that Z3(+)Z3 Γ Z3[x]/(x2).

The above end result classifies the girth of R(+)Z3, and it is fairly shocking that the girth will in no way be four We now think about the state of affairs when M Γ Z2. We will classify when the girth of Γ (R(+)Z2) is three and when it is infinite. We start with the girth three case.

Theorem 2.2. The girth of (R(+)Z2) is three if and solely if one of the following hold:

(i) The girth of (R) is three.

(ii) There exists an r ∈ R∗ such that r2 = 0.

(iii) There exist wonderful a, b ∈ Z(R)∗ such that ab = zero = aZ2 = bZ2.

Proof. (⇐) If (i) holds, the end result is clear. If (ii) holds, word that r · 1 = 0, lest r · (r · 1) = r2 · 1. Then, (r, 0) − (r, 1) − (0, 1) − (r, 0) is a cycle of size three. If (iii) holds, then (a, 0) − (b, 0) − (0, 1) − (a, 0) is a cycle of size 3.

(⇒) Case 1: The aspect (0, 1) is section of a minimal size cycle. Then the cycle has the form (0, 1) − (a, i) − (b, j) − (0, 1). If a = b, we have awesome a, b ∈ Z(R)∗, ab = 0, and aZ2 = bZ2 = 0; if a = b, we have a ∈ R∗ such that a2 = zero.

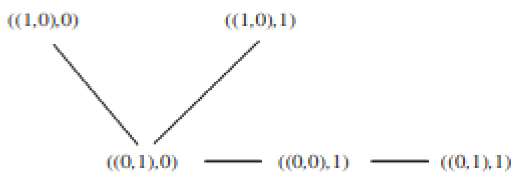
Case 2: The component (0, 1) is now not section of a minimal size cycle. Then, the cycle has the form (a, i) − (b, j) − (c, k) − (a, i). If a, b, and c are all distinct, then a − b − c − a is a cycle in (R), and g((R)) = three If not, then both a2 = zero or b2 = zero We will now supply indispensable and adequate prerequisites for making sure the girth of R(+)Z2 is infinite. We commence with some consequences to be used later.

Lemma 2.3. Let RZ2 × F, the place F is a field. Then, any module operation from R to Z2 is a canonical extension of a module operation both from Z2 to Z2, or from F to Z2 in the case the place F is Z2.

Proof. The annihilator of Z2 as an R-module is an perfect of R; as a consequence ann(Z2) = I1 × I2 = I, where I1 is an best of Z2, and I2 is an best of F. If I1×I2={0}then (1, 0)·1=1=(0, 1)·1, but this would then end result in (1, 1) · 1 = ((1, 0) + (0, 1)) · 1 = (1, 0) · 1 + (0, 1) · 1 = 0, a contradiction. More easily, I1 × I2 R because the module is unitary. Thus, I = {0} × F or I = Z2 × {0}. If I = {0} × F , then the operation is a canonical extension of the module operation from Z2 toZ2. Similarly, if I=Z2×{0}, then the operation is a canonical extension of the module operation from F to Z2.

However, if |F|3, then there is no module operation from F to Z2since there are nonzero sums of devices (which in flip are units), however in the module u · 1 = 1.

Example 2.4. Using Lemma 2.3, let RZ2 × Z2 and think about R(+)Z2. Without loss of generality, the module operation is described by using (0, 0) · 1 = (0, 1) · 1 = zero and (1, 0) · 1 = (1, 1) · 1 = 1. Note that R(+)Z2Z2 × Z2[x]/(x2). Then, g((R(+)Z2)) = ∞, as the zero-divisor design under shows:



Proposition 2.5. Let RZ2 ×F , the place F is a area and |F|3. Then, g((R(+)Z2))=4.

Proof. Since F is a subject and |F|3, by way of Lemma 2.3 the module operation from R to Z2 is an extension of the module operation from Z2 to Z2. We have ((0, 0), 1) − ((0, 1), 0) − ((1, 0), 1) − ((0, a), 0) − ((0, 0), 1) is a cycle of size four (where a ∈ F is nonzero and no equal to 1). By Theorem 2.2, (R(+)Z2) can't incorporate any cycles of size 3, in view that (R) is a megastar layout founded at (1, 0). Hence g((R(+)Z2)) = four

Lemma 2.6. If diam((R)) = 3, then the girth of (R(+)Z2) is finite.

Proof. Let a − b − c − d be a route in (R) with a, b, c, d distinct. If bZ2 = zero andcZ2 = 0, then b · 1 = 1 and c · 1 = 1, however (bc) · 1 = 0, a contradiction. Thus, we must have both bZ2 = zero or cZ2 = 0, or both. Assume bZ2 = zero If cZ2 = 0, then we getthe cycle (b, 0) − (c, 0) − (b, 1) − (c, 1) − (b, 0). If cZ2 = 0, then dZ2 = 0; hence

(b, 0) − (c, 0) − (d, 0) − (c, 1) − (b, 0) is a cycle.

Given the idealization R(+)Z2, it is effortless to see that |R/ann(Z2)| = two Otherwise, let r1+ann(Z2) and r2+ann(Z2) be two cosets awesome from 0+ann(Z2).Thusr1, r2 ∈/ ann(Z2) and so r1 ·1=r2 ·1=1. Therefore (r1−r2) ∈ ann(Z2) and so r1 +ann(Z2) = r2 +ann(Z2).

This end result will be beneficial in the proof of the following.

Theorem 2.7. The girth of Γ (R(+)Z2) is limitless if and solely if R Γ Z2 × Z2 or R is an integral domain.

Proof. (⇐) If R Γ Z2 × Z2, Example 2.4 suggests Γ (R(+)Z2) has no cycles. If R is an

integral domain, then Γ (R(+)Z2) is a famous person plan with middle (0, 1).

(⇒) Lemma 2.6 indicates diam(Γ (R)) 2 or Z(R)∗ = ∅. If Z(R)∗ = ∅, we are done. If

diam(Γ (R)) = 0, then via Theorem 3.2 [1], we have R Γ Z4 or R Γ Z2[x]/(x2). In either

case, there exists a nonzero nilpotent element, and by way of Theorem 2.2, g(Γ (R(+)Z2)) = three If diam(Γ (R)) = 1, then Γ (R) is complete. Thus, if R Γ Z2 × Z2, then R carries a nilpotent

element with the aid of Theorem 2.8 of [2], and by way of Theorem 2.2, g(Γ (R(+)Z2)) = 3, a contradiction.

If diam(Γ (R))=2 and Γ (R)is no longer a famous person graph, then g(Γ (R)) <∞, a contradiction. Thus, by Theorem 2.5 of [2], the solely chances for R are Z2×D, the place D is an necessary domain, or Z(R) is an annihilator ideal. If Z(R) is an annihilator ideal, then R includes a nilpotent element, and we attraction to Theorem 2.2. Hence, R Γ Z2 × D. If |D| = 2, we are done. If D is a finite imperative domain, then D is a discipline and by using Proposition 2.5, g(Γ (R(+)Z2)) = 4, a contradiction. The final case to check out is when D is an limitless indispensable domain.

In R, ann(Z2) is an perfect and subsequently of one of the following three forms: Z2 × {0}, {0} × I, or Z2 × I, the place I is an nonzero perfect of D. If ann(Z2) = Z2 × {0}, then |R/ann(Z2)| > 2 which contradicts the remarks preceeding this result. Again the use of the coset argument, if ann(Z2) = {0} × I or ann(Z2) = Z2 × I, then there exist distinct, nonzero a, b ∈ I such that (0, a), (0, b) ∈ ann(Z2). Thus we structure a cycle ((1, 0), 0) − ((0, a), 0) − ((1, 0), 1) − ((0, b), 0) − ((1, 0), 0). This contradicts g((R(+)Z2)) = ∞.

The following theorem summarizes the outcomes of this section.

Theorem 2.8. Let R be a ring and M an R-module.

(i) g(Γ (R(+)M)) = three if and solely if exactly one of the following hold:

(a) |M| Γ 4,

(b) M Γ Z3 and ann(M) Γ = 0, or

(c) M Γ Z2 and one of the following hold:

(1) g(Γ (R)) = 3,

(2) there exists a nonzero r ∈ R such that r2 = 0, or

(3) there exists wonderful a, b ∈ Z(R)∗ such that ab = zero = aM = bM.

(ii) g(Γ (R(+)M)) = ∞ if and solely if precisely one of the following hold:

(a) M Γ Z3 and ann(M) = zero (and R Γ Z3), or

(b) M Γ Z2 and both R Γ Z2 × Z2 or R is an quintessential domain

We examine that the solely case in which g(Γ (R(+)M)) can be 4 is when M Γ Z2 and R does now not meet any of the above conditions.

For example, via Proposition 2.5 g(Γ (R(+)Z2))= 4 when R Γ Z2 × Z3.

STRUCTURE OF Γ (Z_P^N)

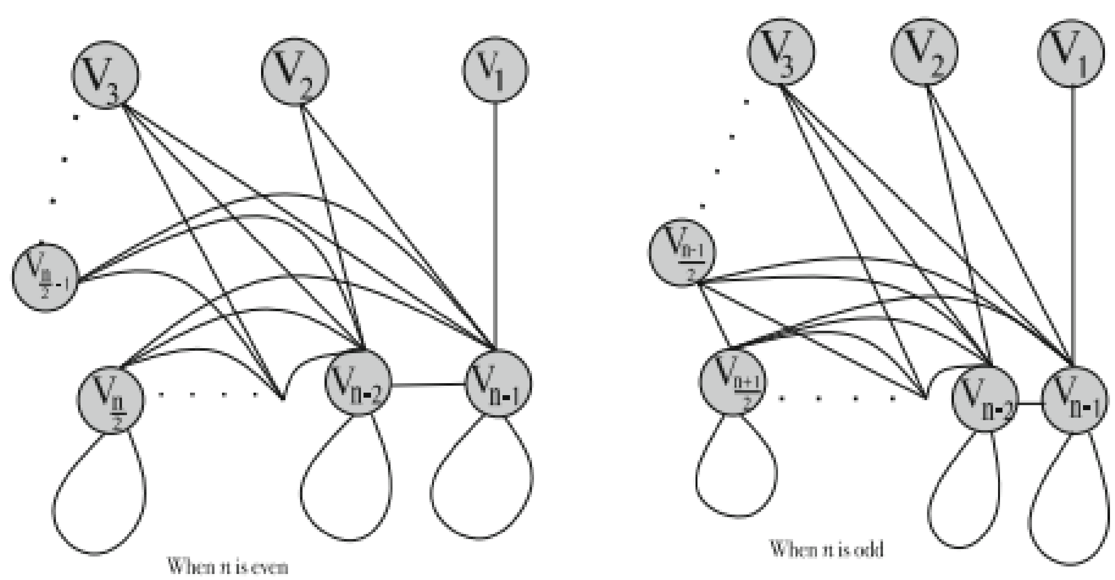


Fig. 1 $\Gamma(\mathbb{Z}_{p^n})$

Let the vertex set of Γ (Zpn) be divided into disjoint subsets V1, V2,..., Vn−1, whereVi = {ki pi : p ki}, 1 ≤ i ≤ n − 1. Then it is no longer challenging to see that

|Vi| = (p − 1)pn−i−1, 1 ≤ i ≤ n − 1 and consequently |Γ (Zpn)| = n−1

i=1

(p − 1)pn−i−1= pn−1 − 1.

The be part of G + H of two vertex-disjoint graphs G and H has vertex set $V(G + H) = V(G) \cup V(H)$ and part set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$

Theorem 1 For a top range p,$\Gamma(Z_{p2}) \sim= K_{p-1}$ and $\Gamma(Z_{p3}) \sim= T_{p(p-1)} + K_{p-1}$, where T_s denotes a definitely disconnected format on s vertices. Proof We first be aware that the vertex set of$\Gamma(Z_{p2})$is given by way of $V = \{1\cdot p, 2\cdot p,...,(p-1)\cdot p\}$. Therefore, x y = zero for all x, $y \in V(\Gamma(Z_{p2}))$. Next, we divide the vertex set of $\Gamma(Z_{p3})$ into two jointly disjoint units $V1 = \{s1 \cdot p : s1 = 1, 2,..., p - 1, p + 1, p + 2,..., 2p - 1, 2p + 1,..., 3p - 1,..., p2 - 1\}$ and $V2 = \{s2 \cdot p2 : s2 = 1, 2,..., p - 1\}$. Therefore, $|V1| = p(p - 1)$ and $|V2| = p - 1$. For any $x1, y1 \in V1$, virtually $x1\ y1 =0$; for all $x2, y2 \in V2$, we have $x2\ y2 = 0$; and for every $z1 \in V1$, $z2 \in V2$, without a doubt $z1z2$ = zero Therefore, $\Gamma(Z_{p3})$ is the be a part of of $T_{p(p-1)}$ and K_{p-1}. Let U, V be subsets of the vertex set of G. Then $U \leftrightarrow V$ shall denote that every vertex of U is adjoining to each vertex of V; and U -V denotes that no vertex of U is adjacent to any vertex of V. A loop at U denotes that all the vertices of U are at the same time adjacent.

Consider the partition of the vertex set of $\Gamma(Z_{pn})$ into the subsets $V1, V2,..., V_{n-1}$, where $V_i = \{k_i\ p_i$: p does no longer divide $k_i\}$, $1 \leq i \leq n-1$. Then $V_i \leftrightarrow V_{n-i}$, for all i = 1, 2,..., n−1 and V_i -V_{n-j} , for all j < i. This system without a doubt offers an algorithm for setting up the zero-divisor sketch of Z_{pn} for all primes p and $n \in N$. This development of the zero-divisor graph of Z_{pn} is explained graphically in Fig. 1. From this building of the zero-divisor graph, we see that every vertex of V_{n-1} represents a minimal dominating set. Thus, the domination quantity of $\Gamma(Z_{pn})$ is 1.

Also, we note that the vertex set of $\Gamma(Z_{pn})$ can be partitioned into two subsets

$X = V1 \cup V2 \cup \cdots \cup V\ n\ 2\ -1$ and $Y = Vn2 \cup Vn2 +1 \cup \cdots \cup V_{n-1}$, the place X is the maximal impartial set and Y induces a clique. With this done, it is now convenient to see that for all pn = 2, 4, we have $rad(\Gamma(Z_{pn})) =1$, $diam(\Gamma(Z_{pn})) = 2$, $\delta(\Gamma(Z_{pn})) = p - 1$, $\Delta(\Gamma(Z_{pn})) = deg(u \in V_{n-1}) = |V1| + |V2| + \cdots + (|V_{n-1}| - 1) = p^{n-1} - 2$, $kv(\Gamma(Z_{pn})) = p - 1$, and $gr(\Gamma(Z_{pn})) = \infty$ if n = 4, 8, 9 3 if otherwise.In the following theorem, we compute the clique range of $\Gamma(Z_{pn})$

Theorem two For a high integer p and $n \geq 4$, the clique quantity of $\Gamma(Z_{pn})$ is equal to Pn 2 − 1 if n is even pn2 if n is odd, the place [x] denotes the biggest integer no longer higher than x.

Proof For any two vertices x, y ∈ V(Γ (Zpn)), and for some k1, k2, (not multiples of p), we have x = k1 pi and y = k2 p j . Then x y = zero if and solely if i + j ≥ n. Thus, for each x ∈ Vt , we have x y = zero and Vt 1 ↔ Vt 2 , for all t, t1, t2 ≥ n2. , the place z denotes the smallest integer now not smaller than z. Therefore, cl(Γ (Zpn)) =|Vt|, t ≥n2 . We first consider the case when n is even. We have cl(Γ (Zpn)) = |Vn−1|+|Vn−2|+···+|Vn2 |= (p − 1)(1 + p + p2 +···+ pn 2 −1) = pn2 − 1.

Now, for the case when n is odd, we have Vt1 ↔ Vt2 if and solely if t1, t2 ≥ [n2] and V[n2] V[2]. Therefore,cl(Γ (Zpn)) =|Vn−1|+|Vn−2|+···+|Vn+12|+ 1= (p − 1)(1 + p + p2 +···+ pn−12 −1)+ 1= pn−12 = p n2.

Let V1, V2,..., Vn−1, the place Vi = {ki pi : p does now not divide ki } be the partition of the vertex set of Γ (Zpn). It is handy to see that all the vertices of Vi have the identical diploma for each i = 1, 2,..., n − 1. If deg(Vi) denotes the diploma of the vertices in Vi , it is effortless to see that deg(Vi) = deg(Vj), for all i = j. With these notations and definitions, the following theorem can be established.

Theorem three For a tremendous integer k, 1 ≤ ok ≤ n−1, the levels of the vertices in Γ (Zpn) are given as deg(Vk) = pk − 1 if 1 ≤ okay < n2 ; pk − two if n2 ≤ ok ≤ n − 1 the place x denotes the smallest integer function.

Proof With the partition V1, V2,..., Vn−1 of the vertex set of Γ (Zpn), we be aware that Vk1 ↔Vk2 if and solely if k1 + k2 ≥ n. We first reflect onconsideration on the case when 1 ≤ ok < n2. Clearly, in this case Vk : Vk and consequently deg(Vk) = |Vn−1|+|Vn−2|+···+|Vn−k | = (p − 1)

1 + p +p2+···+pk−1= pk−1. Next, for n2 ≤ okay ≤ n−1, we note that Vk ↔ Vk for all ok such that n2 ≤ ok ≤ n − 1. Therefore in this case, deg(Vk) = |Vn−1|+|Vn−2|+···+ [|Vk | − 1]+···+|Vn−k | = (p − 1)· 1 + (p − 1)· p + (p − 1)· p2 +···+ (p − 1) ·

pk – 1 +···+(p − 1) · pk−1 = (p − 1)

1 + p + p2 +···+ pk−1

− 1 = (pk − 1) − 1 = pk − two

Corollary 1 Let n ≥ two be a tremendous integer and p be prime. Then the wide variety of edges in Γ (Zpn) is equal to 1

2 [pn−1(np − n − p) − pn−n2 + 2], until n = p = 2.

Proof Let v be a vertex of Γ (Zpn) and let d(v) denote the diploma of v. We first evaluate

v∈V(Γ (Zpn)) d(v). We have

$$
\begin{aligned}
\sum_{v\in V(\Gamma(\mathbb{Z}_{p^n}))} d(v) &= |V_1|deg(V_1) + |V_2|deg(V_2) + \cdots + |V_{\lceil \frac{n}{2}\rceil -1}|deg(V_{\lceil \frac{n}{2}\rceil -1})\\
&\quad + |V_{\lceil \frac{n}{2}\rceil}|deg(V_{\lceil \frac{n}{2}\rceil}) + \cdots + |V_{n-1}|deg(V_{n-1})\\
&= (p-1)p^{n-2}\cdot(p-1) + (p-1)p^{n-3}\cdot(p^2-1) + \cdots + (p-1)p^{n-\lceil \frac{n}{2}\rceil}\\
&\quad \cdot(p^{\lceil \frac{n}{2}\rceil}-1) + (p-1)p^{n-\lceil \frac{n}{2}\rceil -1}\cdot(p^{\lceil \frac{n}{2}\rceil}-2) + \cdots + (p-1)p^0\cdot(p^{n-1}-2)\\
&= (p-1)\left\{(p^{n-1}-1)\cdot p^0 + (p^{n-2}-1)\cdot p^1 + \cdots + (p-1)\cdot p^{n-2}\right\}\\
&\quad -(p-1)\left\{1 + p + \cdots + p^{n-\lceil \frac{n}{2}\rceil -1}\right\}\\
&= (p-1)\left[(n-1)p^{n-1} - \left(1 + p + p^2 + \cdots + p^{n-2}\right)\right]\\
&\quad -(p-1)\left\{1 + p + \cdots + p^{n-\lceil \frac{n}{2}\rceil -1}\right\}\\
&= (p-1)\left[(n-1)p^{n-1} - \left\{\frac{1\cdot(p^{n-1}-1)}{p-1}\right\}\right] - (p-1)\left\{\frac{p^{n-\lceil \frac{n}{2}\rceil}-1}{p-1}\right\}\\
&= p^{n-1}(np-n-p) - p^{n-\lceil \frac{n}{2}\rceil} + 2.
\end{aligned}
$$

Now, since each edge contributes two to the degrees, the number of edges is equal to

$$\frac{1}{2}\left\{\sum_{v\in V(\Gamma(\mathbb{Z}_{p^n}))} d(v)\right\} = \frac{1}{2}[p^{n-1}(np-n-p) - p^{n-\lceil \frac{n}{2}\rceil} + 2]. \qquad \square$$

Theorem 2.10. (1) If Γ(S) is a bipartite graph, then Γ(S) is one of the following graphs: star graph, two-star graph, complete bipartite graph, or complete bipartite graph with a horn.

(2) Every graph of the type given in (1) is the zero-divisor graph of a semigroup with 0.

Proof. (1) This statement follows directly from Proposition 2.10 (2) Any complete bipartite graph is the zero-divisor graph of a semigroup with 0 by [7, Theorem 3(2)] or by [14, Proposition 3.2]. By [8, Theorem 1.3], both a star graph and a two-star graph are zero-divisor graphs of semigroups with 0. Now, it suffices to prove that each complete bipartite graph with a horn is a zero-divisor graph of a

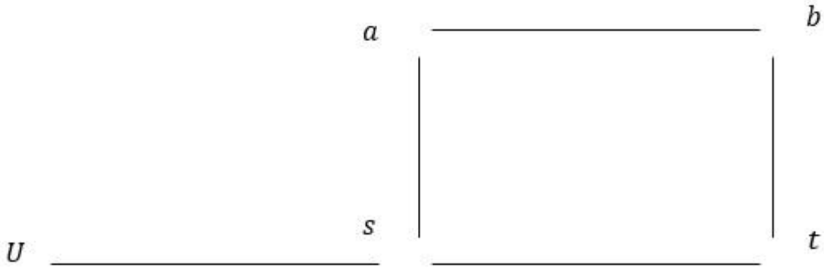


Fig 3: bipartite graph

Semi group with 0. Let G be the graph in Fig. 3. Set S = {0,s, t} ∪ A∪ B ∪U, where s ≠0, t ≠ 0, A∪ B ∪U. Fix a_0 ∈ A and b_0∈ B. Define a commutative binary operation in S by the following:

0S = 0, S^2= sU = 0, $U^{2`}$= {a0}, U(A ∪ {t}) = {a0}, U B = {s},

$$\boldsymbol{Xy} = \begin{cases} a_0 & if\ x, y\ \in\ A\ \cup\ \{t\} \\ b_0 & if\ x, y\ \in\ B \\ 0\ if\ x\ \in\ B\ \cup\ \{s\}, y\ \in\ A\ \cup\ \{t\} \\ s & if\ x\ =\ s, y\ \in\ B \end{cases}$$

Now we check that the associativity holds. Since the operation is commutative, we only need to check $(x\,y)z\ =\ (x\,z)y\ =\ (yz)x\ for\ any\ \{x,y,z\}\ \subseteq\ S.$

$Case\ 1. If\ \{x,y,z\}\ j\ A\ \cup\ \{t\}, then\ (x\,y)z\ =\ (x\,z)y\ =\ (yz)x\ =\ a_0.$

$Case\ 2. If\ \{x,y,z\}\ j\ B, then\ (x\,y)z\ =\ (x\,z)y\ =\ (yz)x\ =\ b_0.$

$Case\ 3. If\ x\ =\ s\ and\ \{y,z\}\ j\ B, then\ (x\,y)z\ =\ (x\,z)y\ =\ (yz)x\ =\ s.$

$Case\ 4. If\ x\ =\ y\ =\ s\ and\ z\ \in\ B, then\ (x\,y)z\ =\ (x\,z)y\ =\ (yz)x\ =\ 0.$

$$Case\ 5. If\ x\ \in\ A\ \cup\ \{t\}, y\ \in\ B\ \cup\ \{s\}, z\ \in\ A\ \cup\ B\ \cup\ \{s,t\}, then\ (x\,y)z\ =\ (x\,z)y$$
$$=\ (yz)x\ =\ 0. Case\ 6. If\ \{x,y,z\}\ j\ U, then\ (x\,y)z\ =\ (x\,z)y\ =\ (yz)x$$
$$=\ a0.$$

$Case\ 7. If\ x \in U, y, z \in A \cup \{t\}, then\ (x\, y)z = (x\, z)y = (yz)x = a_0.$

$Case\ 8. If\ x \in U, y, z \in B, then\ (x\, y)z = (x\, z)y = (yz)x = s.$

$Case\ 9. If\ x \in U, y \in A \cup \{t\}, z \in B, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

$Case\ 10. If\ x \in U, y = s, z \in B, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

$Case\ 11. If\ x \in U, y = s, z \in A \cup \{t\}, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

$Case\ 12. If\ x \in U, y \in U, z \in A \cup \{t\}, then\ (x\, y)z = (x\, z)y = (yz)x = a_0..$

$Case\ 13. If\ x \in U, y \in U, z \in B, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

$Case\ 14. If\ x \in U, y \in U, z = s, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

$Case\ 15. If\ x = y = z = s, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

$Case\ 16. If\ x \in U, y = z = s, then\ (x\, y)z = (x\, z)y = (yz)x = 0.$

Hence S is a zero-divisor semigroup with 0. We easily see that Γ(S) = G and the result follows

2.11 The corresponding zero-divisor semigroups of a class of complete bipartite graphs with a horn In this section, we will determine zero-divisor semigroups whose zero-divisor graphs are as shown in Fig. 6. This also illustrates how the binary operation in the proof of Theorem 2.10 is constructed. Suppose that $S = \{0, s, t, a, b\} \cup U$ is a zero-divisor semigroup with Γ(S) as in Fig. 3. We obtain the following facts for any $u, v \in U$.

(1) $ub = s\ since\ ub \in \overline{N(a)} \cap \overline{N(t)} \cap \overline{N(s)} = \{s\}.$

(2) $s^2 = sub = 0.$

(3) $sb = s\ since\ sb \in \overline{N(u)} \cap \overline{N(a)} \cap \overline{N(t)} = \{s\}.$

(4) $\{a^2, a^2, at\} \subseteq \overline{N(s)} \cap \overline{N(b)} \cup \{0\} = \{a, t, 0\}\ and\ at \neq 0.$

(5) $\{ua, vt\} \subseteq \overline{N(s)} \cap \overline{N(b)} = \{a, t\}.$

$(6)\ uv \in \overline{N(b)} \cap \overline{N(s)} = \{a,t\}\ since\ (uv)b = us = 0.$

$(7)\ b\ 2 = b\ since\ u(b\ 2\ ) = (ub)b = sb = s\ for\ any\ u \in U, and\ this\ implies\ b^2 \neq$
$\in \{0, a, t, s,\} \cup U\ by\ (4)\text{–}(6).$

.	a	t
a	a	a
t	a	a

Table 1

.	a	t
a	0	a
t	a	t

Table 2

.	a	t
a	a	a
t	a	t

Table 3

.	a	t
a	t	a
t	a	t

Table 4

For later convenience, we introduce the following notation.

THE ALGEBRAIC STRUCTURE OF ARTINIAN RINGS

We propose the algebraic composition of Artinian rings. The graph with zero-divisors will exhibit this algebraic structure.

Rule 3.1: Organization Theorem relating to Artinian RingsThe only other finite direct product of Artinian local rings is an Artinian ring R, up to isomorphism.

With the aid of this theorem, we may comprehend the fundamental "building blocks" of Artinian rings, which we shall separate into fields and adjacent rings (that are no longer fields). This study examines the zero-divisor graphs of several neighbourhood rings. After that, we will explore how the zero-divisor graph represents the more challenging underlying algebraic structure by building more challenging Artinian rings using direct products.

Theorem 3.1. Structure Theorem for Artinian Rings : An Artinian ring R is uniquely (up to isomorphism) a finite direct product of Artinian local rings.

This theorem will guide our separation of fields and neighbouring rings, which are essential "building blocks" for Artinian rings (that are now not fields). This essay will look at the zero-divisor graphs for each type of neighbourhood ring. Following that, we'll rent direct merchandise. We will then examine how the zero-divisor strategy presents the more challenging algebraic structure below in order to create more challenging Artinian rings.

Implication : The nilradical Nil(R) in an Artinian ring R is nilpotent and equivalent to the Jacobson Radical.

Proof. We can prove this via induction of n. The case of n = 1 is trivial. It suffices to prove the assertion for n = 2. Let (x1, x2) ∈ I1 × I2 and (r1, r2) ∈ R1 × R2. Then (x1, x2)(r1, r2) = (x1r1, x2r2) ∈ I1 ×I2 and (r1, r2)(x1, x2) = (r1x1, r2x2) ∈ I1 ×I2. So, I1 ×I2 is an ideal of R1 × R2. Now let K ⊆ R1 × R2. Let I1 = {x1 ∈ R1|(x1, x2) ∈ K for some x2 ∈ R2} and I2 = {x2 ∈ R2|(x1, x2) ∈ K for some x1 ∈ R1}. Clearly, K ⊆ I1 × I2. Now let (x1, x2) ∈ I1 × I2. Then (x1, x0 2),(x 0 1 , x2) ∈ K for some x 0 1 and x 0 2 . Then, (x1, x2) = (1, 0)(x1, x0 2) + (0, 1)(x 0 1 , x2) ∈ K. Therefore, K = I1 × I2.

Theorem 3.3. If I = I1 × · · · × In is an ideal of R = R1 × · · · × Rn, then the maximal ideals of R have all I 0 i s equal to Ri for i = 1, . . . , n, except one Ij , with Ij 6= Rj and Ij is maximal.

Proof. From Theorem 3.2., we know that if Ii is an ideal of Ri for i = 1, . . . , n, then I1 × · · · × In is an ideal of R1 × · · · × Rn. Now suppose that I is such that more than one of the Ii ’s is different from Ri . Then we can replace one of these I 0 i s with Ri and get an ideal properly containing I. Therefore, a maximal ideal has all I 0 i s equal to Ri except for one, Ii . It is clear that the one Ii with Ii 6= Ri is also maximal.

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